TWO TOPICS IN THE THEORY OF NONLINEAR SCHRÖDINGER EQUATIONS

by

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Courant Institute of Mathematical Sciences

New York University

May 2018

Pierre Germain

To come out of the Olympic Cinema and be taken aback
by how, in the time it took a dolly to travel
along its little track
to the point where two movie stars' heads
had come together smackety-smack
and their kiss filled the whole screen,

those two great towers directly across the road
at Moy Sand and Gravel
had already washed, at least once, what had flowed
or been dredged from the Blackwater's bed
and were washing it again, load by load,
as if washing might make it clean.

Paul Muldoon, 'Moy Sand and Gravel'

DEDICATION

Dedicated to my parents, Catherine and Pat.

ACKNOWLEDGEMENTS

The single, most significant factor in the advancement of my dissertation research was the dedication of my advisor Pierre Germain. Pierre is singularly committed to the success of his students. He suggests interesting but accessible problems to work on, follows the research process closely, while still giving enough autonomy to develop independent research ability, and does this all with sincere friendliness and enthusiasm. The curse of being such a good advisor is that there are always more students wanting to work with you than you can possibly take. I was one of the lucky ones. Thank you, Pierre.

Among other mathematicians, I would like to thank Zaher Hani, Nader Masmoudi, Miles Wheeler and Chonghung Zeng whose discussions or teaching directly impacted my dissertation research. My close friend and fellow doctoral student Federico Buonerba was a wonderful source of geometric knowledge, and his advice greatly enhanced the research that is presented in Chapter 2. I feel that this is also an appropriate moment to acknowledge the profound impact that my undergraduate mentors Tom Carroll and Steven Wills had on my development as a mathematician.

While research is the primary focus of a Ph.D., teaching is a part of it too – an especially significant part in my case. During my time at NYU I was fortunate to be involved as an educator with the Center for Mathematical Talent, an organization devoted to teaching promising New York City high school students college level mathematics. I am indebted to Mark Saul and, especially, Berna Falay Ok for the opportunity to be a part of their tremendous programming.

Finally, while the work for this dissertation took place in New York City, it would not have been possible without the care, encouragement and strategic independence bestowed on me by my parents in the twenty-three years of my life that I spent in Ireland. It is a myth in America that with enough work you can be whatever you want to be, regardless of where you come from. With Catherine and Pat, though, it may just be true.

Lá Fhéile Pádraig, March 17, 2018,
Brooklyn, New York.

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Introduction

This thesis is composed of two distinct works within the broad field of nonlinear Schrödinger equations. Each of the two chapters begins with an introductory section describing the background and mathematical signficance of the specific topic considered. This introduction knits the chapters together by providing a brief overview of the origins of Schrödinger equations generally. The common theme in the field, and of course in partial differential equations generally, is the influence of physical theories on the trajectory of mathematical research. Both of the topics discussed in this thesis ultimately spring from equations first introduced by physicists to model real-world phenomona.

THE LINEAR SCHRÖDINGER EQUATION

The linear Schrödinger was first introduced by Erwin Schrödinger in 1926 as a model for the behaviour of particles at the atomic level. Neglecting effects due to special relativity, the equation for a single particle reads,

$$i\hbar \frac{\partial u}{\partial t}(x,t) = \left[\frac{-\hbar^2}{2m}\Delta + V(x,t) \right] u(x,t), \tag{0.0.1}$$

where m is the mass, $V(x,t) \in \mathbb{R}$ the potential energy, $\hbar$ is the reduced Plank's constant and the space variable is $x \in \mathbb{R}^d$. The solution of this equation, the *wave function* u, is valued in the complex numbers. Within a year of the publication of the Schrödinger equation, the wave function had been interpreted by Max Born as a generalized probability density. Namely, for future times t, the probability of the particle being positioned in a set $A \subset \mathbb{R}^d$ is,

$$\text{Prob(particle in } A \text{ at time } t) = \int_A |u(x,t)|^2 dx. \tag{0.0.2}$$

Central to this interpretation is the mathematical fact that for any solution u of (0.0.1), the total density $\|u\|_{L^2}^2$ is conserved. Hence if the initial data satisfies $\|u(t=0)\|_{L^2}^2 = 1$, thus defining a probability distribution on $\mathbb{R}^d$ by (0.0.2), for any future time t the solution again defines a probability density by the same formula.

By normalizing the space and time variables one can assume that the constants $\hbar$ and m are 1. There are two especially important cases of the equation. When $V(x,t) = |x|^2$ the equation reads,

$$iu_t(x,t) + \frac{1}{2}\Delta u(x,t) = |x|^2 u(x,t). \tag{0.0.3}$$

This is the equation for the quantum harmonic oscillator. It is an important model because an arbitrary potential V can be approximated by $|x|^2$ in a neighborhood of a minimum (up to translation). Moreover, the equation (0.0.3) can be solved explicitly in terms of the Hermite functions, which form a basis of $L^2(\mathbb{R}^d \to \mathbb{C})$. This method of

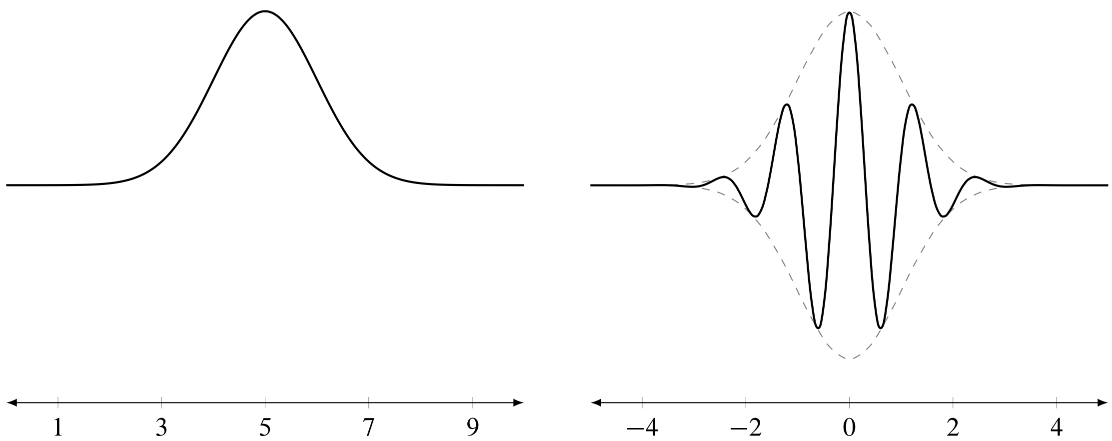

Figure 1: Plots of $\hat{f}(\xi)$ (left) and Re $f(x)$ (right) where f is a Gaussian centered at $\xi = 5$ in Fourier space and $x = 0$ in physical space.

solution is discussed at length in Chapter 1.

When $V(x,t) = 0$, the equation becomes the free Schrödinger equation,

$$iu_t(x,t) + \frac{1}{2}\Delta u(x,t) = 0. \tag{0.0.4}$$

This equation is the starting point for the mathematical theory of Schrödinger equations. It is the preeminent example of a *dispersive partial differential equation.* By dispersive is meant that waves of different frequencies travel at different speeds.

The dispersive character of (0.0.4) can be illustrated by examining the special collection of explicit Gaussian solutions. We will illustrate this for $d = 1$, but the same argument can be performed in higher dimensions. Fix a frequency $\omega \in \mathbb{R}$, a position $a \in \mathbb{R}$, a parameter $\epsilon > 0$ and define a function f by the following formula in Fourier space,

$$\widehat{f}(\xi) = \frac{1}{\sqrt{\pi\epsilon}} \exp\left(-\frac{(\xi-\omega)^2}{2\epsilon^2}\right) \exp(ia\xi).$$

The factor in front ensures that $\|\widehat{f}\|_{L^2} = \|f\|_{L^2} = 1$. Taking the Fourier transform of this Gaussian gives the formula for f in physical space,

$$f(x) = \sqrt{\frac{\epsilon}{\pi}} \exp\left(-\frac{\epsilon^2(x-a)^2}{2}\right) \exp(i\omega(x-a)).$$

This function is a wavelet localized about the frequency $\xi = \omega$ in Fourier space and the position $x = a$ in physical space. The ϵ parameter controls the degree of concentration in Fourier space versus physical space. (By the uncertainty principle, concentration in one comes at the expense of concentration in the other.) Plots of f and $\widehat{f}$ in the case $\omega = 5$, $a = 0$ and $\epsilon = 1$ are given in Figure 1.

The linear Schrödinger equation can be explicitly solved when the initial data is the f. If u satisfies (0.0.4), then the Fourier transform of the unique L^2 solution satisfies $i\hat{u}_t(t,\xi) = (1/2)\xi^2\hat{u}(t,\xi)$. We then find that,

$$\begin{aligned}\hat{u}(\xi,t) &= \exp\left(-\frac{it\xi^2}{2}\right)\frac{1}{\sqrt{\pi\epsilon^2}}\exp\left(-\frac{(\xi-\omega)^2}{2\epsilon^2}\right)\exp\left(ia\xi\right), \\ &= \frac{1}{\sqrt{\pi\epsilon}}\exp\left(-\frac{(1+i\epsilon^2 t)(\xi-\omega)^2}{2\epsilon^2}\right)\exp\left(i(a+t\omega)\xi\right)\right)\exp\left(\frac{i\omega^2 t}{2}\right),\end{aligned}$$

and hence,

$$u(x,t) = \sqrt{\frac{\epsilon}{2\pi(1+i\epsilon^2 t)}}\exp\left(-\frac{\epsilon^2(x-a-t\omega)^2}{2(1+i\epsilon^2 t)}\right)\exp\left(i\omega(x-a-t\omega)\right)\exp\left(\frac{i\omega^2 t}{2}\right).$$

Looking at the magnitude to ignore the phase terms, we have,

$$|u(x,t)| = \sqrt{\frac{\epsilon}{\pi\sqrt{1+\epsilon^2 t^2}}}\exp\left(-\frac{\epsilon}{2(1+\epsilon^2 t^2)}(x-a-t\omega)^2\right).$$

There are two key observations to make.

- The Gaussian is travelling at speed ω. That is, the wave is traveling at a speed given by its frequency.

  This implies that if one begins the flow with a superposition of two waves centered at the same point in physical space, but with different frequencies, they will travel at different speeds and spread apart. Namely, they will disperse.

  Note in particular that the group velocity is independent of a and ϵ.

- The Gaussian is spreading out as time increases, and in the limit as $t \to \infty$, $u(x,t)$ goes to 0 pointwise. This is really a consequence of the first observation. The Gaussian is itself the superposition of many frequencies. The components of the solution corresponding to these frequencies are moving at different speeds, causing the solution to spread out.

  When the wave is highly concentrated in Fourier space (that is, when ϵ is small), the wave spreads out slowly. This is because the frequencies are concentrated in a small region in Fourier space, and the superpositioned components of the solution are thus travelling at comparable speeds. Conversely, when ϵ is large the solution is the superposition of a wide set of frequencies and the pieces are traveling at quite different speeds.

Using the fundamental solution of the free Schrödinger equation,

$$u(x,t) = \frac{1}{(2\pi i t)^{d/2}}\int_{\mathbb{R}^d} e^{i|x-y|^2/(2t)} f(y)dy,$$

one can quantify the cumulative dispersive effect by the following decay inequality,

$$\|x \mapsto u(x,t)\|_{L^\infty} \leq \frac{\|f\|_{L^1}}{(2\pi)^{d/2}t^{d/2}}. \tag{0.0.5}$$

In particular, for L^1 initial data, the solution converges pointwise to 0 as $t \to \infty$. The L^1 norm appearing as the convergence constant is consistent with the explicit Gaussian solutions we found. One easily calculates that $\|f\|_{L^1} = \sqrt{2/\epsilon}$, so when ϵ is small (large concentration in Fourier space) the constant factor in (0.0.5) is large, and the decay is slower.

The decay inequality (0.0.5) can be used to prove an enormously useful family of estimates known as the Strichartz estimates. These capture the dispersive character of the equation in a set of space-time norms. To state these estimates, and for our discussion in the following subsections, we let $e^{it\Delta}u_0$ denote the solution of $iu_t + \Delta u = 0$ with initial data u_0. (In what follows it is easier to normalize the space variable again to remove the $1/2$ constant in front of the Laplacian, as we do from now on.)

Theorem (Strichartz estimates for the free Schrödinger equation)**.** *Fix $d \geq 1$. Call a pair of exponents (q, r) admissible if $2 \leq q, r \leq \infty$, $2/q + d/r = d/2$, and $(q, r, d) \neq (2, \infty, 2)$.*

- *For any admissible pair of exponents (q, r), there holds the homogeneous Strichartz estimate,*

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r} \leq C(d, q, r)\|u_0\|_{L_x^2}. \tag{0.0.6}$$

- *For any admissible pairs of exponents (q, r) and (a, b), there holds the inhomogeneous Strichartz estimate,*

$$\left\| \int_0^t e^{(t-s)\Delta} F(s) ds \right\|_{L_t^q L_x^r} \leq C(d, q, r, a, b)\|F\|_{L_t^{a'} L_x^{b'}}, \tag{0.0.7}$$

  where a' and b' denote the Hölder conjugate of a and b respectively.

See [43] for a comprehensive discussion of these Strichartz estimates as well as a proof of the main cases. We will use these estimates in Chapter 2 to prove a global critical wellposedness result for a certain nonlinear Schrödinger equation. On the other hand, one of the consequences of our work in Chapter 1 is a proof of the estimate (0.0.6) in the homogeneous case $(d, q, r) = (1, 6, 6)$. Our proof moreover reveals the best constant in this case to be $C(1, 6, 6) = (1/12)^{1/12}$.

Nonlinear Schrödinger equations

A number of nonlinear equations related to the linear Schrödinger equation appear in physical theories. The two topics discussed in this thesis both spring from two such nonlinear equations. The first equation we discuss is the nonlinear Schrödinger equation (NLS),

$$iu_t + \Delta u = \mu|u|^2 u, \tag{0.0.8}$$

where $\mu \in \mathbb{R}$. This specific NLS arises in the theory of nonlinear optics. It is a special case of the general NLS equation,

$$iu_t + \Delta u = \mu|u|^{p-1} u, \tag{0.0.9}$$

where $\mu \in \mathbb{R}$ and $p > 1$.

An enormous amount of mathematical research has been carried out on (0.0.9). In the original form of the equation, the space variable is $\mathbb{R}^d$, and so the initial data is a map from $\mathbb{R}^d$ to $\mathbb{C}$. It is also possible to pose the equation where the space variable is $\mathbb{T}^d$ or another domain. The dynamics of the equation are hugely dependent on the domain chosen. Below we show that when the domain is $\mathbb{R}^d$, the dispersive property of the linear equation can be used (essentially by itself) to prove global small-data wellposedness. On the other hand, when the domain is $\mathbb{T}^d$, the linear equation exhibits dispersion but this does not translate into decay because of the geometry of the torus, as we will see. Understanding the dynamics here is extremely difficult. The theory of continuous resonant equations arose as a means of advancing research on NLS in this case.

First, we will show by example how the decay estimates for the linear Schrödinger equation can be used to prove global critical wellposedness for NLS when the space domain is $\mathbb{R}^d$. The theorem is for the L^2 critical case, namely when the equation is invariant under the L^2 scaling $f(x) \mapsto \lambda^{d/2} f(\lambda x)$. The principal ingredient in the proof is the decay property of the linear equation, quantified using the Strichartz estimates.

Theorem 0.0.1 (Critical L^2 NLS solutions)**.** *Let p be the L^2 critical exponent $p = 1 + 4/d$ and let $\mu = \pm 1$. There exists $\epsilon > 0$ such that if $\|u_0\|_{L^2_x} \leq \epsilon$ then there exists a unique global solution of NLS* (0.0.9) *in $C^0_t L^2_x$.*

Observe that when $d = 2$ this theorem is precisely for the cubic NLS (0.0.8). The proof we present here follows closely the proof given in [43].

Proof. The fixed point Duhamel representation of (0.0.9) is,

$$u(t) = T[u(t)] := e^{it\Delta} u_0 - i\mu \int_0^t e^{i(t-s)\Delta} \left(|u(s)|^{p-1} u(s) \right) ds. \tag{0.0.10}$$

In order to prove wellposedness we show that T is a contraction mapping in the space-time function space S^0 defined by the norm,

$$\|u\|_{S^0} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q_t L^r_x}.$$

Note that $C^0_t L^2_x \subset S^0$ as $(\infty, 2)$ is always an admissible exponent pair.

Let (q,r) and (a,b) be admissible exponent pairs in the sense of the Strichartz estimates. We have, using (0.0.10) and both Strichartz estimates (0.0.6), (0.0.7),

$$\|Tu(t)\|_{L^q_t L^r_t} \lesssim \|e^{it\Delta} u_0\|_{L^q_t L^r_x} + \||u(t)|^p\|_{L^{a'}_t L^{b'}_x} \lesssim \|u_0\|_{L^2_x} + \|u(t)\|^p_{L^{pa'}_t L^{pb'}_x}.$$

Under the assumptions $p = 1 + 4/d$ and (a,b) being admissible, we find that (pa', pb') is also admissible. Hence,

$$\|Tu(t)\|_{S^0} \lesssim \|u_0\|_{L^2_x} + \|u\|^p_{S^0}.$$

It is standard that if ϵ is sufficiently small, this inequality implies that T maps an ϵ ball of S^0 centered at 0 to itself.

We next prove that T is a contraction on a sufficiently small ball around 0. Let $v(t), u(t) \in S^0$ and again let (q,r) and (a,b) be admissible exponent pairs. By the inhomogeneous Strichartz estimate (0.0.7), we have,

$$\begin{aligned} \|Tv(t) - Tu(t)\|_{L^q_t L^r_x} &= \left\| \int_0^t e^{i(t-s)\Delta} \left[|u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s) \right] ds \right\|_{L^q_t L^r_x} \\ &\lesssim \left\| |u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s) \right\|_{L^{a'}_t L^{b'}_x} \\ &\lesssim \left(\|u\|^{p-1}_{L^{pa'}_t L^{pb'}_x} + \|v\|^{p-1}_{L^{pa'}_t L^{pb'}_x} \right) \|u - v\|_{L^{pa'}_t L^{pb'}_x}, \end{aligned}$$

where we have used the pointwise estimate $\left| |z_1|^{p-1} z_1 - |z_2|^{p-1} z_2 \right| \leq p(|z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2|$. Again, (pa', pb') is admissible, so taking supremums we have,

$$\|Tv(t) - Tu(t)\|_{S^0} \lesssim \left(\|u\|^{p-1}_{S^0} + \|v\|^{p-1}_{S^0} \right) \|u - v\|_{S^0}.$$

Hence by choosing the ϵ ball sufficiently small, T is a contraction. □

The overall conclusion is that if we have decay, we can access the dynamics. Now, by comparison, consider

NLS posed for maps on the torus. The linear equation is,

$$iu_t(x,t) + \Delta u(x,t) = 0, \tag{0.0.11}$$

for $x \in \mathbb{T}^d$. By expanding the solution $u(x,t)$ in a time-dependent Fourier series, we find that it is possible to explicitly solve the PDE. When we do this we find that the solution u is quasi-periodic.

For concreteness, consider the case $d = 1$. We can identify maps from the circle $\mathbb{T}$ to $\mathbb{C}$ with periodic functions from $[0, 2\pi]$ to $\mathbb{C}$. Expanding such a function $u : [0, 2\pi] \to \mathbb{C}$ as a Fourier series,

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(t)e^{inx},$$

and substituting into (0.0.11), we find that the coefficient functions satisfy $ia_n'(t) - n^2 a_n(t) = 0$. Hence $a_n(t) = e^{-in^2t}a_n(0)$. The linear Schrödinger equation is thus explicitly solvable with solution,

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n(0)e^{-in^2t+inx}. = \sum_{n\in\mathbb{Z}} a_n(0)e^{in(x-nt)}.$$

The component of the solution corresponding to the discrete frequency n is still traveling at speed n, and hence in this sense there is still dispersion. However the solution is periodic in time, with period 2π, so there cannot possibly be decay. Intuitively, the problem is in the geometry: waves travel apart, but because of the structure of the torus these waves eventually meet again.

In all, the kind of global Strichartz estimates that were central in the proof above cannot hold. Continuous resonant equations were introduced in [19] as a way of studying the dynamics of NLS on the torus. The central idea is to approximate the NLS in a certain time and size regime by a different PDE, the continuous resonant equation, for which we can say more about the dynamics. These dynamics are then projected back to NLS through a rigorous approximation theorem. Chapter 1 of this thesis continues this research direction by studying two continuous resonant type equations set on $\mathbb{R}$.

Schrödinger maps equations

The Landau-Lipshitz-Gilbert equation (LLG) is another nonlinear Schrödinger equation that appears in physics. It arises in the physical theory of ferromagnetism. In its simplest form it reads,

$$u(x,t) \times u_t(x,t) = \Delta u(x,t) + |\nabla u(x,t)|^2 u(x,t), \tag{0.0.12}$$

where the function u is a map from $\mathbb{R}^d$ to $\mathbb{R}^3$ and $\times$ is the cross product of vectors in $\mathbb{R}^3$. The solution is actually valued in the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$: given initial data $u_0 : \mathbb{R}^d \to \mathbb{R}^3$ satisfying $|u_0(x)| = 1$ pointwise, one can show for smooth solutions that $|u(x,t)| = 1$ for all x and t. The equation is thus an equation for maps valued in the sphere.

It is not immediately obvious that (0.0.12) is a Schrödinger equation at all. To see that it is, we make three observations. First, as an equation involving derivatives of a function valued on $\mathbb{S}^2$, the equation should be understood as an equality involving elements of $T_u\mathbb{S}^2$. Recall that the tangent space $T_u\mathbb{S}^2$ is explicitly given by,

$$T_u\mathbb{S}^2 = \{v \in \mathbb{R}^3 \mid \langle v, u\rangle_{\mathbb{R}^3} = 0\}.$$

Second, for $v \in T_u\mathbb{S}^2$ we calculate that $u \times (u \times v) = -v$. This means that the operator $Jv = u \times v$ satisfies

$J^2 = -I$ and thus behaves similarly to scalar multiplication by i (specifically the property $i^2 = -1$). J is called a *complex structure* for this reason. Finally, because $u_x \in T_u\mathbb{S}^2$, we have $\langle u, u_x\rangle = 0$ and hence,

$$0 = \frac{d}{dx}\langle u, u_x\rangle = |u_x|^2 + \langle u, u_{xx}\rangle.$$

This gives that the projection of u_{xx} onto the tangent space is given by,

$$P(u_{xx}) = u_{xx} - \langle u_{xx}, u\rangle = u_{xx} + |u_x|^2 u.$$

The equation (0.0.12) thus directly generalizes the linear Schrödinger equation to maps valued on the sphere: multiplication by i is replaced by multiplication by the complex structure J, and the Laplacian Δu is replaced by the spherical Laplacian $\Delta u + |\nabla u|^2 u$.

The LLG equation motivates the notion of Schrödinger maps, which is the topic discussed in Chapter 2. Schrödinger maps generalize the usual linear Schrödinger equation (0.0.11) and the LLG equation (0.0.12) to arbitrary maps $u : M \to N$ where M is a Riemannian manifold and N is a complex manifold. The Schrödinger maps equation reads, $Ju_t = \Delta_{M\to N} u$, where J is the complex structure on N and $\Delta_{M\to N}$ is the Laplace-Beltrami operator for maps from M to N.

Research on Schrödinger maps has largely focused on the special case (0.0.12), both because of its physical relevance and its mathematical accessibility. Chapter 2 of this thesis is motivated by the desire to study global features of Schrödinger maps when the complex dimension of N is greater than 1. By studying the special case when N is complex projective space, and making an equivariant ansatz, we are able to determine a mathematically accessible equation describing the dynamics and access some of the higher dimensional theory.

Chapter 1

Continuous Resonant Equations

§1.1 · INTRODUCTION

In recent years resonant systems have emerged as extremely useful tools for studying nonlinear Schrödinger equations (NLS). Resonant equations have been used to construct solutions of the cubic NLS on $\mathbb{T}^2$ that exhibit large growth of Sobolev norms [15]. They have appeared as modified scattering limits for a number of equations, including the cubic NLS on $\mathbb{R} \times \mathbb{T}^d$ [31], the cubic NLS on $\mathbb{R}^d$ with $2 \leq d \leq 5$ and harmonic trapping in all but one direction [32], and a coupled cubic NLS system on $\mathbb{R} \times \mathbb{T}$ [45]. The continuous resonant equation (CR) was originally shown to approximate the dynamics of small solutions of the two-dimensional cubic NLS on a large torus $\mathbb{T}_L^2$ over long times scales (longer than L^2/ϵ^2, where ϵ is the size of the initial data) [19]. Recent work has extended this by showing that a whole family of CR equations approximate the dynamics of NLS on $\mathbb{T}_L^d$ for arbitrary dimension and arbitrary analytic nonlinearity [12]. The original two-dimensional cubic CR equation is the same resonant system that appears in the modified scattering limit in [32] for $d = 3$; it has also been shown to be a small data approximation for the cubic NLS with harmonic trapping set on $\mathbb{R}^2$ [25].

One of the principal reasons that resonant systems are useful is that they generally exhibit a large amount of structure. They are often Hamiltonian and usually possess many symmetries, a good wellposedness theory, and an infinite number of orthogonal, explicit solutions. Extensive work has been done on studying such purely dynamical properties of the CR equations: starting in the paper that introduced the original two-dimensional cubic equation [19], in subsequent works again on this cubic case [24, 25], and a more recent paper on the general case [11]. This research fits into a larger program of studying the dynamics of nonlocal Hamiltonian PDEs; we mention, for example, work on the Szegö equation [22] and the lowest Landau level equation [21].

The two-dimensional cubic CR equation has, in particular, been found to have many remarkable dynamical properties. The PDE is symmetric under many non-trivial actions such as the Fourier transform and the linear flow of the Schrödinger equation (with or without harmonic trapping); it is Hamiltonian, and through these symmetries admits a number of conserved quantities. The equation is globally wellposed in L^2 and all higher Sobolev spaces. It has many explicit stationary wave solutions, including all of the Hermite functions and the function $1/|x|$. All stationary waves that are in L^2 are automatically analytic and exponentially decaying in physical space and Fourier space.

The work of the present chapter was initiated by the question of whether these striking properties also hold for the only other continuous resonant equation that scales like L^2: the one-dimensional quintic continuous resonant equation. Our investigation subsequently broadened to include another one-dimensional resonant equation that is somewhat more physically relevent, and turns out to be the modified scattering limit in [32] for $d = 2$. Our overall finding is that these Hamiltonian systems do display much of the remarkable dynamical structure of the two-dimensional cubic CR. In fact, we are able to show that both systems belong to a large class of Hamiltonian systems on the phase space $L^2(\mathbb{R} \to \mathbb{C})$, and that each system is this class bears many of the features of L^2 critical CR: they have a strong symmetry structure, global wellposedness in L^2 and other Sobolev spaces, , and many explicit stationary wave solutions in the form of the Hermite functions. Typical members of the class lack much of the structure of both cubic two-dimensional and quintic one-dimensional CR – for example, it is not the case that all L^2 stationary waves are analytic – but our findings do suggest that a number of the properties of the L^2 critical CR equations are generic.

1.1.1 · Presentation of the equations

The two systems we study in this chapter are resonant systems corresponding to the nonlinear Schrödinger equation with harmonic trapping,

$$iu_t - \Delta u + x^2 u = iu_t + Hu = |u|^{2k} u, \tag{1.1.1}$$

where the spatial variable is $x \in \mathbb{R}$ and $k = 1, 2$ is an integer, so that the nonlinearity is analytic. The cubic $k = 1$ equation is physically relevant: in this case, (1.1.1) is the Gross–Pitaevskii equation and is a model in the physical theory of Bose–Einstein condensates [29].

Let us first see how the resonant equations arise. Looking at the profile $v(t) = e^{-itH} u(t)$ (where e^{itH} is the propagator of the linear equation $iu_t + Hu = 0$), we find it satisfies,

$$iv_t = e^{-itH} \left(|e^{itH} v|^{2k} e^{itH} v \right).$$

Expressing $v(t)$ in the basis of eigenfunctions of the operator H (namely the Hermite functions), the equation on v can be written as,

$$iv_t(t) = \sum_{n_1,\dots,n_{2k+2} \in \mathbb{Z}^+} e^{2itL} \Pi_{n_{2k+2}} \left[\prod_{m=1}^{k} \left((\Pi_{n_m} v(t)) (\overline{\Pi_{n_{k+1+m}} v(t)}) \right) \Pi_{n_{k+1}} v(t) \right], \tag{1.1.2}$$

where $\Pi_n v$ is the projection onto the eigenspace of H corresponding to eigenvalue $2n + 1$. The phase L in (1.1.2) is given by $L = n_1 + \ldots + n_{k+1} - (n_{k+2} + \ldots + n_{2k+2})$. The resonant terms in the sum in (1.1.2) are the terms that are not oscillating in time; that is, those satisfying $L = 0$. The resonant system corresponding to (1.1.2) is obtained by considering only the resonant terms; namely,

$$iw_t(t) = \sum_{\substack{n_1,\dots,n_{2k+2} \in \mathbb{Z}^+ \\ L=0}} \Pi_{n_{2k+2}} \left[\prod_{m=1}^{k} \left((\Pi_{n_m} w(t)) (\overline{\Pi_{n_{k+1+m}} w(t)}) \right) \Pi_{n_{k+1}} v(t) \right]. \tag{1.1.3}$$

We will show in Section 1.2 that this resonant PDE may be written more compactly in terms of a certain time average of the nonlinearity,

$$iw_t(t) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{-isH} \left(|e^{isH} w(t)|^{2k} e^{isH} w(t) \right) ds. \tag{1.1.4}$$

From this expression we are able to infer that the resonant system is, up to a rescaling of time, the Hamiltonian flow on the phase space $L^2(\mathbb{R} \to \mathbb{C})$ corresponding to the Hamiltonian,

$$\mathcal{H}_{2k+2}(f) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |(e^{itH} f)(x)|^{2k+2} dx dt. \tag{1.1.5}$$

The overall resonant program is to gain information on the dynamics of solutions to (1.1.1) by studying the associated resonant system (1.1.4). This program has two, distinct components. The first is to establish approximation results that rigorously demonstrate that solutions of the resonant system well approximate solutions of the full system in certain function spaces and over certain timescales. The second component of the program is to understand the dynamics of the resonant equation itself. One then projects these dynamics back to the original equation through the approximation results.

We start the chater by proving an approximation result that is valid for all positive integers k. We then analyze the resonant system (1.1.4) in depth for the cubic case, when $k = 1$, and the quintic case, when $k = 2$. These two cases are particularly significant for separate reasons.

- The cubic case $k = 1$ is physically relevant, as previously mentioned. In addition, the resonant equation here is exactly the resonant equation obtained in [32] as the modified scattering limit of the NLS equation,

$$iu_t - u_{xx} - u_{yy} + |y|^2 u = |u|^2 u, \tag{1.1.6}$$

  where the space variable is $(x, y) \in \mathbb{R}^2$. Precisely, consider small initial data $u_0(x, y)$. Suppose that $u(x, y, t)$ solves (1.1.6) with initial data $(x, y) \mapsto u_0(x, y)$. For each fixed x, let $w(x, y, t)$ be the solution of the resonant equation (1.1.3) with initial data $y \mapsto u_0(x, y)$. Then,

$$\lim_{t\to\infty} \left\| u(x, y, t) - e^{it(-\partial_{xx} - \partial_{yy} + |y|^2)} w(x, y, 2\ln(t)) \right\|_{H^N(\mathbb{R}^2)} = 0,$$

  where H^N is the usual Sobolev space. (This holds for any $N \geq 8$ so long as the initial data is sufficiently small.)

- In the quintic case, $k = 2$, we will prove that the resonant system (1.1.4) is precisely the one-dimensional quintic continuous resonant equation. It is the only CR equation, other than the original two-dimensional cubic CR equation, that scales like L^2. One of the central motivations of this work is to understand the dynamics of the CR system in this important special case.

1.1.2 · OBTAINED RESULTS

1.1.2.1 · An approximation theorem

We begin, in Section 1.2, by proving the following theorem, which shows that solutions of the resonant equation (1.1.4) well-approximate solutions of the full equation (1.1.1) on a long time scale. This theorem is essentially a lower dimensional version of Theorem 3.1 in [25], and our proof that proof closely.

Theorem (Theorem 1.2.3, page 20)**.** *Define the space $\mathcal{H}^s$ by the norm $\|f\|_{\mathcal{H}^s} = \|H^{s/2} f\|_{L^2}$; this is equivalent to the norm $\|\langle x\rangle^s f\|_{L^2} + \|\langle\xi\rangle^s \widehat{f}\|_{L^2}$. Fix $s > 1/2$ and initial data $u_0 \in \mathcal{H}^s$. Let u be a solution of the nonlinear Schrödinger equation with harmonic trapping* (1.1.1) *and w a solution of the resonant equation* (1.1.4), *both corresponding to the* *initial data u_0. Suppose that the bounds $\|u(t)\|_{\mathcal{H}^s}, \|w(t)\|_{\mathcal{H}^s} \leq \epsilon$ hold for all $t \in [0,T]$. Then for all $t \in [0,T]$,*

$$\|u(t) - e^{itH} w(t)\|_{\mathcal{H}^s} \leq \left(t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1}\right) \exp\left((2k+1)t\epsilon^{2k}\right).$$

In particular if $t \lesssim \epsilon^{-2k}$ then $\|u(t) - e^{itH} w(t)\|_{\mathcal{H}^s} \lesssim \epsilon^{2k+1}$.

1.1.2.2 · *Representation formulas for the Hamiltonians*

Following the approximation result, we focus on studying the resonant system (1.1.4) in the cases $k = 2$ and $k = 1$. Right away we note that the Hamiltonians (1.1.5) for these systems arise from multilinear functionals, in the following way. In the quintic case ($k = 2$, so that the nonlinearity in (1.1.4) is order 5) the Hamiltonian is,

$$\mathcal{H}_6(f) = \frac{2}{\pi} \|e^{itH} f\|_{L_t^6 L_x^6}^6 = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |(e^{itH} f)(x)|^6 dx dt, \tag{1.1.7}$$

which arises from the multilinear functional,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} (e^{itH} f_1)(e^{itH} f_2)(e^{itH} f_3)\overline{(e^{itH} f_4)(e^{itH} f_5)(e^{itH} f_6)} dx dt, \tag{1.1.8}$$

through $\mathcal{H}_6(f) = \mathcal{E}_6(f, f, f, f, f, f)$. The cubic Hamiltonian, corresponding to $k = 1$, is,

$$\mathcal{H}_4(f) = \frac{2}{\pi} \|e^{itH} f\|_{L_t^4 L_x^4}^4 = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |(e^{itH} f)(x)|^4 dx dt, \tag{1.1.9}$$

which is associated to the multilinear functional,

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} (e^{itH} f_1)(e^{itH} f_2)\overline{(e^{itH} f_3)(e^{itH} f_4)} dx dt, \tag{1.1.10}$$

through $\mathcal{H}_4(f) = \mathcal{E}_4(f, f, f, f)$. The fact that the Hamiltonians can be expressed in terms of multilinear functionals is a nontrivial structural property that guides much of the analysis. The symmetries of the Hamiltonian, its wellposedness theory and the existence of certain stationary wave solutions can all be determined from studying the associated multilinear functional (see Theorems 1.3.8 and 1.3.10 for examples of this in practice). We will prove that Hamilton's equations corresponding to $\mathcal{H}_6$ and $\mathcal{H}_4$ are given by,

$$i u_t = \mathcal{T}_6(u, u, u, u, u) \quad \text{and} \quad i u_t = \mathcal{T}_4(u, u, u),$$

respectively, where the multilinear operators $\mathcal{T}_6$ and $\mathcal{T}_4$ are defined implicitly by the formulas,

$$\langle \mathcal{T}_6(f_1, f_2, f_3, f_4, f_5), g \rangle_{L^2} = 6\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, g),$$

and,

$$\langle \mathcal{T}_4(f_1, f_2, f_3), g \rangle_{L^2} = 4\mathcal{E}_4(f_1, f_2, f_3, g).$$

Hamilton's equations are precisely the resonant equations (1.1.4) in the cases $k = 2$ and $k = 1$.

In the study of resonant equations, it has turned out to be fundamental to determine alternative representations for the Hamiltonian $\mathcal{H}$, the associated multilinear functional $\mathcal{E}$ and associated multilinear operator $\mathcal{T}$. These alternative representations often reveal structure that is concealed by specific representations such as (1.1.7) and (1.1.9). In Sections 1.4 and 1.5 we derive numerous representations for $\mathcal{E}_6$ and $\mathcal{E}_4$ respectively. First, for $\mathcal{E}_6$, we find the two formulas,

$$\begin{aligned}\mathcal{E}_6(f_1,f_2,f_3,f_4,f_5,f_6) &= \frac{2}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)\overline{(e^{it\Delta}f_4)(e^{it\Delta}f_5)(e^{it\Delta}f_6)}dxdt \qquad (1.1.11)\\ &= \frac{1}{\pi^2}\int_{\mathbb{R}^6} f_1(y_1)f_2(y_2)f_3(y_3)f_4(y_4)f_5(y_5)f_6(y_6)\\ &\qquad\qquad \delta_{y_1+y_2+y_3=y_4+y_5+y_6}\delta_{y_1^2+y_2^2+y_3^2=y_4^2+y_5^2+y_6^2}dy,\end{aligned}$$

where, in the first equation, $e^{it\Delta}$ denotes the propagator of the linear Schrödinger equation. These representations both show that the quintic Hamiltonian system is the one-dimensional quintic continuous resonant equation [12].

To describe our next representations, we require some notation. For an isometry $A:\mathbb{R}^3\to\mathbb{R}^3$, let E_A be the multilinear functional,

$$E_A(f_1,f_2,f_3,f_4,f_5,f_6) = \int_{\mathbb{R}^3} f_1((Ax)_1)f_2((Ax)_2)f_3((Ax)_3)\overline{f_4(x_1)f_5(x_2)f_6(x_3)}dx_1dx_2dx_3, \qquad (1.1.12)$$

where $(Ax)_k = \langle Ax, e_k\rangle$. The functional E_A is a special case of the type of functional that appears in Brascamp–Lieb inequalities [10]. We then have the following representations: for the quintic equation, we prove that,

$$\mathcal{E}_6(f_1,f_2,f_3,f_4,f_5,f_6) = \frac{1}{2\sqrt{3}\pi^2}\int_0^{2\pi} E_{R(\theta)}(f_1,f_2,f_3,f_4,f_5,f_6)d\theta, \qquad (1.1.13)$$

where $R(\theta)$ is the rotation of $\mathbb{R}^3$ by θ radians about the axis $(1,1,1)$; while for the cubic equation, we prove that,

$$\mathcal{E}_4(f_1,f_2,f_3,f_4) = \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} E_{S(\theta)}(G,f_1,f_2,G,f_3,f_4)d\theta, \qquad (1.1.14)$$

where $G(x) = e^{-x^2/2}$, and $S(\theta)$ is the rotation of $\mathbb{R}^3$ by θ radians about the axis $(0,1,1)$;

The two representations (1.1.13) and (1.1.14) are extremely beneficial for studying $\mathcal{H}_6$ and $\mathcal{H}_4$. They also place the two Hamiltonians in a larger class of Hamiltonians that, we will find, share much of the same structure. This is not obvious: *a priori* we might expect the Hamiltonians $\mathcal{H}_6$ and $\mathcal{H}_4$ to be quite unalike. The differences in (1.1.13) and (1.1.14) are also of note. The presence of the Gaussians G in (1.1.14) ultimately causes the symmetry group of the cubic equation to be smaller than that of the quintic equation; it is also prevents the cubic equation from having a scaling law, which has consequences for the possible stationary waves we can construct.

1.1.2.3 · Properties of a class of multilinear functionals

Formulas (1.1.13) and (1.1.14) suggest that one can learn much about the dynamics of the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$ by studying the class of functionals E_A. This is precisely what we do in Section 1.3. Our overall finding is that many of the remarkable properties of the two-dimensional cubic CR equation may be found at the level of the functionals E_A. By (1.1.13) and (1.1.14), these properties are inherited directly by $\mathcal{E}_6$ and $\mathcal{E}_4$. We will show that the functionals E_A have a large group of symmetries, that the associated PDE problem is locally wellposed in every Sobolev space and globally wellposed in L^2, and that all of the Hermite functions are stationary wave solutions of the associated PDE problem. We emphasize that A is always assumed to be an isometry of $\mathbb{R}^3$, as it is in (1.1.13) and (1.1.14).

Theorem (Theorems 1.3.2 and 1.3.3, page 25). *The functional E_A is invariant under the following actions (for any λ):*

(i) Fourier Transform: $f_k \mapsto \widehat{f}_k$. *(iv) Quadratic modulation:* $f_k \mapsto e^{i\lambda x^2} f_k$.
(ii) Modulation: $f_k \mapsto e^{i\lambda} f_k$. *(v) Schrödinger group:* $f_k \mapsto e^{i\lambda\Delta} f_k$.
(iii) L^2 scaling: $f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x)$. *(vi) Schrödinger with harmonic trapping:* $f_k \mapsto e^{i\lambda H} f_k$.

If, in addition, A satisfies $A(1,1,1) = (1,1,1)$ (as is the case for $R(\theta)$ in (1.1.13)*), then E_A is invariant under the following actions (for any λ):*

(vii) Linear modulation: $f_k \mapsto e^{i\lambda x} f_k$. *(viii) Translation:* $f_k \mapsto f_k(\cdot + \lambda)$.

All of these actions give symmetries for $\mathcal{H}_6$, by (1.1.13). For $\mathcal{H}_4$, the symmetries are not inherited automatically because of the Gaussian terms in (1.1.14), however we find that five of the eight symmetries do hold.

Corollary (Theorem 1.4.8, page 42, and Theorem 1.5.5, page 61). *The functional $\mathcal{E}_6$ is invariant under all the actions (i) through (viii). The functional $\mathcal{E}_4$ is invariant under the actions (i), (ii), (vi), (vii) and (viii).*

These symmetries, seen purely at the level of E_A, are used to generate conserved quantities for the resonant equations, using Noether's Theorem.

Corollary (Table 1.4.2, page 43, and Table 1.5.2, page 62). *The following are conserved quantities of the resonant equation* (1.1.4) *in the cases $k = 2$ and $k = 1$,*

$$\int_{\mathbb{R}} |f(x)|^2 dx, \qquad \int_{\mathbb{R}} x|f(x)|^2 dx, \qquad \int_{\mathbb{R}} if'(x)\overline{f}(x)dx, \qquad \int_{\mathbb{R}} |xf(x)|^2 + |f'(x)|^2 dx.$$

In the quintic case $k = 2$, we have the additional conserved quantities,

$$\int_{\mathbb{R}} \left[ixf'(x) + f(x)\right]\overline{f}(x)dx, \qquad \int_{\mathbb{R}} |xf(x)|^2 dx, \qquad \int_{\mathbb{R}} |f'(x)|^2 dx.$$

We next examine the L^2 boundedness of E_A.

Theorem (Theorem 1.3.6, page 29). *There holds the bound,*

$$|E_A(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \prod_{k=1}^{6} \|f_k\|_{L^2}.$$

In particular, $|H_A(f)| := |E_A(f,f,f,f,f,f)| \leq \|f\|_{L^2}^6$. If A is not a signed permutation, there is equality in the Hamiltonian bound if and only if f is a Gaussian.

This bound is actually an example of a geometric Brasscamp-Lieb inequality, and the classification of the maximizers is already known [2]. We prove the inequality and classify the maximizers in our case in a way that appears to be new.

The L^2 bound on E_A directly gives L^2 bounds for $\mathcal{H}_6$ and $\mathcal{H}_4$; indeed,

$$|\mathcal{H}_6(f)| \leq \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} |E_{R(\theta)}(f,f,f,f,f,f)| d\theta \leq \frac{1}{\sqrt{3}\pi} \|f\|_{L^2}^6,$$

and similarly, $|\mathcal{H}_4(f)| \leq (1/\sqrt{2\pi})\|f\|_{L^2}^4$. There is equality in these only if f is a Gaussian. We find that for $\mathcal{H}_6$ any Gaussian gives equality, whereas for $\mathcal{H}_4$ not every Gaussian does, essentially because of the lack of a scaling law; see Proposition 1.5.7.

Using the representation $\mathcal{H}_6(f) = (2/\pi)\|e^{it\Delta} f\|_{L_t^6 L_x^6}$, from (1.1.11), the L^2 bound on $\mathcal{H}_6$ reads,

$$\|e^{it\Delta} f\|_{L_t^6 L_x^6}^6 \leq \frac{1}{2\sqrt{3}}\|f\|_{L^2}^6,$$

which is the homogeneous Strichartz inequality in dimension one. Our work shows that the constant here is the best possible, and that there is equality if and only if f is a Gaussian. These facts were previously determined in [20].

We then turn to the PDE problem associated to E_A. Given E_A, a multilinear operator T_A is defined implicitly by,

$$\begin{aligned}\langle T_A(f_1, \ldots, f_5), g\rangle = 2E_A(f_1, f_2, f_3, g, f_4, f_5) + 2E_A(f_1, f_2, f_3, f_4, g, f_5) \\ + 2E_A(f_1, f_2, f_3, f_4, f_5, g).\end{aligned}$$

From the representations (1.1.13) and (1.1.14), we find representations for the resonant equations,

$$iu_t = \mathcal{T}_6(u, u, u, u, u) = \frac{1}{\sqrt{3}\pi^2}\int_0^{2\pi} T_{R(\theta)}(u, u, u, u, u)d\theta, \tag{1.1.15}$$

and,

$$iu_t = \mathcal{T}_4(u, u, u) = \frac{1}{\sqrt{2}\pi^2}\int_0^{2\pi} T_{S(\theta)}(G, u, u, G, u)d\theta, \tag{1.1.16}$$

where $R(\theta)$ and $S(\theta)$ are the same matrices as in (1.1.13) and (1.1.14).

Theorem (Theorem 1.3.7, page 30). *The mutilinear operator T_A is bounded from X^5 to X for (i) $X = L^2$, (ii) $X = L^{2,\sigma}$ for any $\sigma \geq 0$, and (iii) $X = H^\sigma$ for any $\sigma \geq 0$.*

This theorem automatically implies analogous bounds for $\mathcal{T}_6$ and $\mathcal{T}_4$, and this leads directly to local wellposedness for the resonant equations in all the spaces in the theorem. By pairing this local wellposedness result with the conservation of the L^2 norm, we get global wellposedness in L^2.

Corollary (Theorems 1.4.12, page 45, and 1.5.9, page 64). *Hamilton's equations corresponding to $\mathcal{H}_6$ and $\mathcal{H}_4$ are locally wellposed in X for (i) $X = L^2$, (ii) $X = L^{2,\sigma}$ for any $\sigma \geq 0$, and (iii) $X = H^\sigma$ for any $\sigma \geq 0$. They are globally wellposed in L^2.*

Finally, we find that the functional E_A interacts well with the Hermite functions.

Theorem (Theorem 1.3.9, page 33). *Let $\{\phi_n\}_{n=0}^\infty$ be the Hermite functions. If $n_1 + n_2 + n_3 \neq n_4 + n_5 + n_6$, then $E_A(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}) = 0$. It follows that,*

$$T_A(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}) = C\phi_{n_6},$$

for some C and $n_6 = n_1 + n_2 + n_3 - n_4 - n_5$.

By using the representations of $\mathcal{T}_6$ and $\mathcal{T}_4$ in (1.1.15) and (1.1.16), and the fact that $G = C\phi_0$, we immediately discover that,

$$\mathcal{T}_6(\phi_n, \phi_n, \phi_n, \phi_n, \phi_n) = C_n\phi_n \quad \text{and} \quad \mathcal{T}_4(\phi_n, \phi_n, \phi_n) = D_n\phi_n,$$

for some constants C_n and D_n. This immediately implies that $e^{-iC_nt}\phi_n(x)$ and $e^{-iD_nt}\phi_n(x)$ are explicit solutions of the resonant equations (1.1.15) and (1.1.16) respectively. We recall that a solution of the form $e^{i\omega t}\psi(x)$ is a *stationary wave solution.*

Corollary. *For every $n \geq 0$, $\phi_n(x)$ is a stationary wave of the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$.*

By letting the symmetries of each of the equations act on ϕ_n we can construct more stationary waves; see (1.4.37) and (1.5.24).

1.1.2.4 · The quintic Hamiltonian, $\mathcal{H}_6$

The previous subsection outlined results on the quintic Hamiltonian $\mathcal{H}_6$ that all arise from the representation (1.1.7) along with relevant properties of the functional E_A. Such results also apply to any composite Hamiltonian of the form,

$$\mathcal{H}(f) = \int_\Omega \phi(\omega) E_{A(\omega)}(f, \ldots, f) d\omega, \tag{1.1.17}$$

where $A(\omega)$ is always an isometry and ϕ is integrable. One of the aspirations of the present work is that other Hamiltonian systems may be cast into the framework of (1.1.17), and that our results on the functional E_A may be applied therein.

The Hamiltonian $\mathcal{H}_6$, however, has more structure than a generic Hamiltonian of type (1.1.17). In Section 1.4 we present a number of results that are based on this additional structure and that do not follow simply from analogous properties of E_A. We first prove that if a stationary wave is in L^2, then it is automatically analytic and exponentially decaying in physical space and Fourier space.

Theorem (Corollary 1.4.16, page 50). *Suppose that $\phi \in L^2$ is a stationary wave solution of the quintic resonant equation* (1.1.3). *Then there is $\alpha, \beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\hat{\phi} e^{\beta x^2} \in L^\infty$. In particular, ϕ can be extended to an analytic function on the complex plane.*

We then investigate further boundedness properties of $\mathcal{E}_6$, which lead directly to local wellposedness of Hamilton's equation in the relevant spaces. Our first result is that $\mathcal{T}_6$ is smoothing: it maps Sobolev data to a higher Sobolev space. The second result concerns boundedness in weighted L^∞ spaces.

Theorem. *(i) (Theorem 1.4.17, page 50) For any $\sigma > 0$, there is a $\delta > 0$ and a constant C such that,*

$$\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{2,\sigma+\delta}} \leq C \prod_{k=1}^{5} \|f_k\|_{L^{2,\sigma}},$$

and,

$$\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{H^{\sigma+\delta}} \leq C \prod_{k=1}^{5} \|f_k\|_{H^\sigma}.$$

(ii) (Theorem 1.4.18, page 53) For any $s > 1/2$ there is a constant C such that,

$$\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{\infty,s}} \leq C \prod_{k=1}^{5} \|f_k\|_{L^{\infty,s}}.$$

It is expected that item (ii) here can be sharpened to show that $\mathcal{T}_6$ is bounded from $(\dot{L}^{\infty,1/2})^5$ to $\dot{L}^{\infty,1/2}$ (note these are homogeneous weighted L^∞ spaces). As discussed after the proof of Theorem 1.4.18, this is equivalent to $1/\sqrt{|x|}$ being a stationary wave of the quintic Hamiltonian system, which we conjecture.

1.1.2.5 · The cubic Hamiltonian, $\mathcal{H}_4$

As in the quintic case, we present a number of results on the cubic resonant equation that rely on further structure of $\mathcal{H}_4$ beyond that given by the representation (1.1.16). Again, we prove that stationary waves are analytic and exponentially decaying in physical space and Fourier space once they are in L^2, and we examine boundedness in weighted L^∞ spaces. Our stationary waves theorem, and the broad plan of the proof, are the same as those of the quintic equation, but the technical details are quite different.

Theorem (Theorem 1.5.13, page 68)**.** *Suppose that $\phi \in L^2$ is a stationary wave solution of the cubic resonant equation* (1.1.3)*. Then there is $\alpha, \beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\hat{\phi} e^{\beta x^2} \in L^\infty$. In particular, ϕ can be extended to an analytic function on the complex plane.*

Theorem. *For any $s > 1/2$ there is a constant C such that, $\|\mathcal{T}_4(f_1, f_2, f_3)\|_{L^{\infty,s}} \leq C \prod_{k=1}^{3} \|f_k\|_{L^{\infty,s}}$.*

1.1.3 · PLAN OF THE CHAPTER

In Section 1.2 we prove the main approximation result. Section 1.3 is devoted to studying the functionals E_A defined in (1.1.12). We study the functionals in somewhat more generality than indicated above: we assume the matrix A is an isometry from $\mathbb{R}^n$ to $\mathbb{R}^n$ and, then, that E_A takes $2n$ inputs. In Section 1.4 we present results concerning the quintic Hamiltonian system defined by $\mathcal{H}_6$, including the representation formulas and the details of how our findings on E_A translate to $\mathcal{E}_6$. The cubic Hamiltonian system is treated in a similar fashion in Section 1.5. There is one appendix that deals with the technical classification of the maximizers of the L^2 bound on E_A.

1.1.4 · NOTATIONS AND CONVENTIONS

- For $x \in \mathbb{R}$, the Japanese bracket is $\langle x \rangle = \sqrt{1 + x^2}$.
- $\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(x)\overline{g}(x)dx$.
- The Sobolev space H^σ is defined by the norm $\|f\|_{H^\sigma} = \|\langle x \rangle^\sigma \widehat{f}\|_{L^2}$.
- The weighted space $L^{2,\sigma}$ is defined by the norm $\|f\|_{L^{2,\sigma}} = \|\langle x \rangle^\sigma f\|_{L^2}$.
- $H = -\Delta + x^2$ is the operator corresponding to the quantum harmonic oscillator.
- The Fourier transform of f is $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$. With this convention, the map $f \mapsto \hat{f}$ is an isometry of $L^2(\mathbb{R})$, and the identity $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$ holds. We will frequently use the Fourier inversion formula,
$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ia\langle w,x \rangle} \phi(w)dwdx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{\phi}(aw)dw = \frac{1}{|a|}\phi(0). \tag{1.1.18}$$
- We set $G(x) = e^{-x^2/2}$. For all $a > 0$, $\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right)(\xi) = a^{-1/2} e^{-\frac{\xi^2}{2a}}$ and $\int_{\mathbb{R}} e^{-ax^2} dx = \sqrt{\pi/a}$.
- $A \lesssim B$ means there is an absolute constant C such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

§1.2 · AN APPROXIMATION THEOREM

We begin the chapter by treating more precisely the derivation of the resonant equation (1.1.4) and then proving the approximation theorem described in the introduction.

Before studying the nonlinear problem, we recall some basic properties of the linear problem corresponding to (1.1.1). These facts will be used extensively throughout the chapter. The linear equation corresponding to (1.1.1) is simply the equation for the quantum harmonic oscillator,

$$iu_t + Hu = iu_t - \Delta u + x^2 u = 0, \tag{1.2.1}$$

where $H = -\Delta + x^2$. For any initial data $u_0 \in L^2$ there is a unique solution to (1.2.1), which we denote $e^{itH}u_0$. An explicit representation of this solution is given by the *Mehler formula*,

$$e^{itH}u_0(x) = \frac{1}{\sqrt{2\pi|\sin(2t)|}} \int_{\mathbb{R}} e^{-i\left[(x^2/2+y^2/2)\cos(2t)-xy\right]/\sin(2t)} u_0(y)dy. \tag{1.2.2}$$

(This and other properties of the linear flow may be found in [13].) From this expression we see that the solution is time-periodic with period π.

An alternative representation of the solution of (1.2.1) may be found by examining the *Hermite functions* $\{\phi_n\}_{n=0}^{\infty}$. The Hermite functions are eigenfunctions of H – they satisfy $H\phi_n = (2n+1)\phi_n$ – and they form an orthonormal basis of L^2. Each of these functions is a polynomial multiplied by the Gaussian $e^{-x^2/2}$; for example,

$$\phi_0(x) = c_0 e^{-x^2/2}, \qquad \phi_1(x) = c_1 x e^{-x^2/2}, \qquad \phi_2(x) = c_2(1-2x^2)e^{-x^2/2},$$

where the constants c_n are normalizing constants that ensure $\|\phi_n\|_{L^2} = 1$. Using the eigenfunction property one finds that $e^{itH}\phi_n = e^{it(2n+1)}\phi_n$. Let $\Pi_n u_0 = \langle u_0, \phi_n\rangle \phi_n$ be the orthogonal projection onto the eigenspace spanned by ϕ_n. Given any $u_0 \in L^2$ we may expand $u_0(x) = \sum_{n=0}^{\infty}(\Pi_n u_0)(x)$, and then find,

$$e^{itH}u_0(x) = \sum_{n=0}^{\infty} e^{it(2n+1)}(\Pi_n u_0)(x),$$

so the flow has a simple description in the Hermite function coordinates. We finally note that the Hermite functions satisfy $\phi_n(-x) = (-1)^n\phi_n(x)$, as may be infered from the formula $\phi_n(x) = c_n e^{x^2/2}(d^n/dx^n)e^{-x^2}$ from [13].

We now turn to the nonlinear problem (1.1.1). The linear part of the equation may be absorbed into the nonlinearity by changing variables to the profile $v(x,t) = e^{-itH}u(x,t)$. The function v satisfies the equation,

$$iv_t = e^{-itH}\left(|e^{itH}v|^{2k}e^{itH}v\right) := N_t(v,\dots,v), \tag{1.2.3}$$

where N_t is the $(2k+1)$ multilinear functional,

$$N_t(f_1,\dots,f_{2k+1}) = e^{-itH}\left[\left(\prod_{m=1}^{k}(e^{itH}f_m)(\overline{e^{itH}f_{k+1+m}})\right)(e^{itH}f_{k+1})\right]. \tag{1.2.4}$$

We expand each of the functions f_m in the basis of Hermite functions,

$$e^{itH} f_m = e^{itH} \left(\sum_{n_m=0}^{\infty} \Pi_{n_m} f_m \right) = \sum_{n_m=0}^{\infty} e^{it(2n_m+1)} \Pi_{n_m} f_m,$$

and then substitute into (1.2.4). This yields,

$$N_t(f_1, \ldots, f_{2k+1}) = \sum_{n_1,\ldots,n_{2k+2}\geq 0} e^{2iLt} \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} f_m)(\overline{\Pi_{n_{k+1+m}} f_{k+1+m}}) \right) \Pi_{n_{k+1}} f_{k+1} \right], \tag{1.2.5}$$

where $L = \sum_{m=1}^{k+1} n_m - n_{k+m+1}$.

In (1.2.5), when $L \neq 0$ the associated term in the sum is oscillating, while when $L = 0$ the associated term is not. The resonant equation arises simply from neglecting the oscillatory terms. Define the multilinear functional $\mathcal{T}$ by

$$\mathcal{T}(f_1, \ldots, f_{2k+1}) = \sum_{\substack{n_1,\ldots,n_{2k+2}\geq 0 \\ L=0}} \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} f_m)(\overline{\Pi_{n_{k+1+m}} f_{k+1+m}}) \right) \Pi_{n_{k+1}} f_{k+1} \right]. \tag{1.2.6}$$

The resonant PDE is then given by,

$$i w_t = \mathcal{T}(w, \ldots, w). \tag{1.2.7}$$

Lemma 1.2.1. *The resonant functional $\mathcal{T}$ is the time average of the functionals N_r over the interval $[-\pi/4, \pi/4]$; that is,*

$$\mathcal{T}(f_1, \ldots, f_{2k+1}) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} N_r(f_1, \ldots, f_{2k+1}) dr. \tag{1.2.8}$$

Proof. We integrate the sum in (1.2.5) over $[-\pi/4, \pi/4]$ term by term. If $L = 0$ nothing changes and we get the associated term in (1.2.6). If L is even then $\int_{-\pi/4}^{\pi/4} e^{2iLr} dr = 0$, and the term in (1.2.5) is 0. Finally if L is odd, then either n_{2k+2} is even and $L - n_{2k+2}$ is odd, or n_{2k+2} is odd and $L - n_{2k+2}$ is even. In the first case we have, using the Hermite function property $(\Pi_n f)(-x) = (-1)^n (\Pi_n f)(x)$, that,

$$\begin{aligned} &\left(\prod_{m=1}^{k} ((\Pi_{n_m} f_m)(-x))(\overline{(\Pi_{n_{k+1+m}} f_{k+1+m})(-x)}) \right) (\Pi_{n_{k+1}} f_{k+1})(-x) \\ &\qquad = (-1)^{L-n_{2k+2}} \left(\prod_{m=1}^{k} ((\Pi_{n_m} f_m)(x))(\overline{(\Pi_{n_{k+1+m}} f_{k+1+m})(x)}) \right) (\Pi_{n_{k+1}} f_{k+1})(x), \end{aligned}$$

and hence the function here is odd. Projecting onto the eigenspace spanned by the even function $\phi_{n_{2k+2}}$ gives the 0 vector. The associated term in the sum (1.2.5) is thus 0. In the case when n_{2k+2} is even and $L - n_{2k+2}$ is odd a similar analysis shows that the term in the sum is again 0. In conclusion, all of terms corresponding to $L \neq 0$ vanish, while those corresponding to $L = 0$ are unchanged. □

By virtue of the lemma the resonant equation can be written as,

$$\begin{aligned} i w_t = \mathcal{T}(w, \dots, w) &= \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} N_r(w(t), \dots, w(t)) dr \\ &= \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} e^{-irH} \left(|e^{irH} w(t)|^{2k} e^{irH} w(t) \right) dr, \end{aligned} \tag{1.2.9}$$

which is precisely (1.1.4). One can show that the resonant equation is, up to a rescaling of time, the flow corresponding the Hamiltonian,

$$\mathcal{H}_{2k+2}(f) = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} |e^{irH} f(x)|^{2k+2} dx dr.$$

The details of this Hamiltonian correspondence are presented in Theorem 1.4.1 below.

We now prove the approximation theorem. The theorem is essentially a lower dimensional analog of Theorem 3.1 in [25], and our proof follows theirs closely. The function space in our theorem is,

$$\mathcal{H}^s = \{u \in L^2 : H^{s/2} u \in L^2\},$$

with the norm $\|u\|_{\mathcal{H}^s} = \|H^{s/2} u\|_{L^2}$. From [46], we have the norm equivalence,

$$\|u\|_{\mathcal{H}^s} \sim \|\langle x \rangle^{s/2} u\|_{L^2} + \|\langle \xi \rangle^{s/2} \hat{u}\|_{L^2}.$$

This space $\mathcal{H}^s$ is useful for two reasons: first, if $s > 1/2$, then the space is an algebra (as a direct consequence of the norm equivalence); and, second, the space interacts well with the linear propagator e^{itH}, as seen in the following Lemma.

Lemma 1.2.2. *Fix $s \geq 0$. For all $u \in \mathcal{H}^s$ and $t \in \mathbb{R}$ we have $\|e^{itH} u\|_{\mathcal{H}^s} \leq \|u\|_{\mathcal{H}^s}$.*

A general L^p version of this lemma appears in [8]; for L^2, there is the following shorter proof.

Proof. First let s be an even non-negative integer. Write $u \in L^2$ in the basis of Hermite functions as $u = \sum_{n=0}^{\infty} a_n \phi_n$. Becuase s is a non-negative even integer, for every n we have $H^{s/2} \phi_n = (2n+1)^{s/2} \phi_n$. This then gives,

$$\begin{aligned} \|e^{itH} u\|_{\mathcal{H}^s}^2 &= \left\| H^{s/2} \sum_{n=0}^{\infty} a_n e^{it(2n+1)} \phi_n \right\|_{L^2}^2 = \left\| \sum_{n=0}^{\infty} a_n e^{it(2n+1)} (2n+1)^{s/2} \phi_n \right\|_{L^2}^2 \\ &= \sum_{n=0}^{\infty} |a_n e^{it(2n+1)} (2n+1)^{s/2}|^2 \|\phi_n\|_{L^2}^2 \quad \text{(by orthogonality)} \\ &= \sum_{n=0}^{\infty} |a_n (2n+1)^{s/2}|^2 \|\phi_n\|_{L^2}^2 = \|u\|_{\mathcal{H}^s}^2. \end{aligned}$$

The result for general s follows from interpolation. □

Theorem 1.2.3. *Fix $s > 1/2$ and initial data $u_0 \in \mathcal{H}^s$. Let u be a solution of the nonlinear Schrödinger equation with harmonic trapping* (1.1.1) *and w a solution of the resonant equation* (1.2.7)*, both corresponding to the initial* *data u_0. Suppose that the bounds $\|u(t)\|_{\mathcal{H}^s}, \|w(t)\|_{\mathcal{H}^s} \le \epsilon$ hold for all $t \in [0,T]$. Then for all $t \in [0,T]$,*

$$\|u(t) - e^{itH} w(t)\|_{\mathcal{H}^s} \le \left(t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1}\right) \exp\left((2k+1)t\epsilon^{2k}\right).$$

In particular if $t \lesssim \epsilon^{-2k}$ then $\|u(t) - e^{-itH} w(t)\|_{\mathcal{H}^s} \lesssim \epsilon^{2k+1}$.

Proof. Let $v(x,t) = e^{-itH}u(x,t)$, so that v satisfies the PDE (1.2.3). We note that $v(x,0) = u(x,0) = u_0(x)$. Using the lemma, we find that,

$$\|u(t) - e^{itH} w(t)\|_{\mathcal{H}^s} = \|e^{itH} v(t) - e^{itH} w(t)\|_{\mathcal{H}^s} \le \|v(t) - w(t)\|_{\mathcal{H}^s}. \tag{1.2.10}$$

To prove the theorem it therefore suffices to show that v and w are close in $\mathcal{H}^s$.

Therefore let v and w be solutions of the equations (1.2.3) and (1.2.7) respectively with the same initial data u_0,

$$i v_t(t) = N_t(v(t), \dots, v(t)) = e^{-itH}\left(|e^{itH} v(t)|^{2k} e^{itH} v(t)\right), \tag{1.2.11}$$

$$i w_t(t) = \mathcal{T}(w(t), \dots, w(t)) = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} e^{-irH}\left(|e^{irH} w(t)|^{2k} e^{irH} w(t)\right) dr, \tag{1.2.12}$$

$$u_0(x) = v(x,0) = u(x,0). \tag{1.2.13}$$

Set,

$$D_t(f_1, \dots, f_{2k+1}) = N_t(f_1, \dots, f_{2k+1}) - \mathcal{T}(f_1, \dots, f_{2k+1}) \tag{1.2.14}$$

$$= \sum_{\substack{n_1,\dots,n_{2k+2} \ge 0 \\ L \ne 0}} e^{2itL} \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} f_m)\overline{(\Pi_{n_{k+1+m}} f_{k+1+m})}\right) \Pi_{n_{k+1}} f_{k+1}\right]. \tag{1.2.15}$$

From the expressions of the multilinear operators N_t and $\mathcal{T}$ in (1.2.11) and (1.2.12) (or their multilinear versions (1.2.4) and (1.2.8)), from Lemma 1.2.2, and from the fact that $\mathcal{H}^s$ is an algebra, it follows that N_t and $\mathcal{T}$ are uniformly bounded from $(\mathcal{H}^s)^{2k+1}$ to $\mathcal{H}^s$. The same holds for D_t from (1.2.14).

Set $\phi(t) = v(t) - w(t)$. Because $\phi(0) = 0$, the Duhamel form of the equation on ϕ is,

$$i\phi(t) = \int_0^t \left[\mathcal{T}(v(r), \dots, v(r)) - \mathcal{T}(w(r), \dots, w(r)) + D_r(v(r), \dots, v(r))\right] dr. \tag{1.2.16}$$

We will determine *a priori* bounds on ϕ. For the first term in the integrand here, we can expand by multilinearity to find,

$$\begin{aligned}
&\|\mathcal{T}(v(r), \dots, v(r)) - \mathcal{T}(w(r), \dots, w(r))\|_{\mathcal{H}^s} \\
&\qquad \le \sum_{m=0}^{2k} \|\mathcal{T}(\underbrace{v(r), \dots, v(r)}_{m \text{ times}}, v(r) - w(r), \underbrace{w(r), \dots, w(r)}_{2k-m \text{ times}})\|_{\mathcal{H}^s} \\
&\qquad \le \sum_{m=0}^{2k} \|v(r)\|_{\mathcal{H}^s}^m \|v(r) - w(r)\|_{\mathcal{H}^s} \|w(r)\|_{\mathcal{H}^s}^{2k-m} \\
&\qquad \le (2k+1)\epsilon^{2k} \|v(r) - w(r)\|_{\mathcal{H}^s}.
\end{aligned} \tag{1.2.17}$$

For the second term in the integrand in (1.2.16) we need to look more closely at the operator D_t. We first observe the identity,

$$e^{2irL} = \frac{d}{dr} \int_{\frac{\pi}{2}\lfloor \frac{2r}{\pi} \rfloor}^{r} e^{2i\theta L} d\theta,$$

where $\lfloor x \rfloor$ is the smallest integer less that x. (Recall from the proof of the first lemma that only even values of L contribute to the sum in (1.2.15).) The interval of integration here has length less than 1. We can then handle the second term in (1.2.16) as follows,

$$\begin{aligned}
&\int_0^t [D_r(v(r), \ldots, v(r))]\, ds \\
&= \sum_{\substack{n_1,\ldots,n_{2k+2}\geq 0 \\ L\neq 0}} \int_0^t e^{2irL} \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} v(r))(\overline{\Pi_{n_{k+1+m}} v(r)}) \right) \Pi_{n_{k+1}} v(r) \right] dr \\
&= \sum_{\substack{n_1,\ldots,n_{2k+2}\geq 0 \\ L\neq 0}} \int_0^t \frac{d}{dr} \left(\int_{\frac{\pi}{2}\lfloor \frac{2r}{\pi} \rfloor}^{r} e^{2i\theta L} d\theta \right) \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} v)(\overline{\Pi_{n_{k+1+m}} v}) \right) \Pi_{n_{k+1}} v \right] dr.
\end{aligned}$$

Using integration by parts, we have,

$$\begin{aligned}
&\text{(left hand side)} \\
&= - \sum_{\substack{n_1,\ldots,n_{2k+2}\geq 0 \\ L\neq 0}} \int_0^t \left(\int_{\frac{\pi}{2}\lfloor \frac{2r}{\pi} \rfloor}^{r} e^{2i\theta L} d\theta \right) \frac{d}{dr} \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} v)(\overline{\Pi_{n_{k+1+m}} v}) \right) \Pi_{n_{k+1}} v \right] dr \\
&\qquad + \sum_{\substack{n_1,\ldots,n_{2k+2}\geq 0 \\ L\neq 0}} \left(\int_{\frac{\pi}{2}\lfloor \frac{2t}{\pi} \rfloor}^{t} e^{2i\theta L} d\theta \right) \Pi_{n_{2k+2}} \left[\left(\prod_{m=1}^{k} (\Pi_{n_m} v)(\overline{\Pi_{n_{k+1+m}} v}) \right) \Pi_{n_{k+1}} v \right] \\
&= - \int_0^t \sum_{m=0}^{2k} \int_{\frac{\pi}{2}\lfloor \frac{2r}{\pi} \rfloor}^{r} D_\theta(\underbrace{v(r), \ldots, v(r)}_{m \text{ times}}, v_r(r), \underbrace{v(r), \ldots, v(r)}_{2k-m \text{ times}}) d\theta dr \\
&\qquad + \left(\int_{\frac{\pi}{2}\lfloor \frac{2t}{\pi} \rfloor}^{t} D_\theta(v(t), \ldots, v(t)) d\theta \right).
\end{aligned}$$

Because the interval of integration $\left[\frac{\pi}{2}\left\lfloor \frac{2t}{\pi} \right\rfloor, t\right]$ has length less than 1, we get,

$$\begin{aligned}
\left\| \int_0^t [D_r(v(r), \ldots, v(r))]\, ds \right\|_{\mathcal{H}^s} &\leq t(2k+1) \sup_{r\in[0,t]} \left(\|v(r)\|_{\mathcal{H}^s}^{2k} \|v_r(r)\|_{\mathcal{H}^s} \right) + \|v(t)\|_{\mathcal{H}^s}^{2k+1} \\
&\leq t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1},
\end{aligned} \tag{1.2.18}$$

where in the last line we have used $\|v_t\|_{\mathcal{H}^s} \leq \|v\|_{\mathcal{H}^s}^{2k+1} \leq \epsilon^{2k+1}$, coming from (1.2.3).

Combining the estimates (1.2.17) and (1.2.18) we get

$$\|\phi(t)\|_{\mathcal{H}^s} \leq (2k+1)\epsilon^{2k} \left(\int_0^t \|\phi(s)\|_{\mathcal{H}^s} ds \right) + t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1}.$$

Gronwell's inequality then implies that,

$$\|v(t) - w(t)\|_{\mathcal{H}^s} = \|\phi(t)\|_{\mathcal{H}^s} \leq \left(t(2k+1)\epsilon^{4k+1} + \epsilon^{2k+1} \right) \exp\left((2k+1)t\epsilon^{2k} \right),$$

which, with (1.2.10), gives the Theorem. □

§1.3 · ANALYSIS OF A CLASS OF MULTILINEAR FUNCTIONALS

In the introduction we presented two formulas (1.1.7) and (1.1.9) that represent the Hamiltonians $\mathcal{H}_6$ and $\mathcal{H}_4$ in terms of simpler functionals of the form E_A,

$$\mathcal{H}_6(f) = \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_{R(\theta)}(f, f, f, f, f, f) d\theta, \tag{1.3.1}$$

$$\mathcal{H}_4(f) = \frac{1}{2\sqrt{2}\pi^2} \int_0^{2\pi} E_{S(\theta)}(G, f, f, G, f, f) d\theta. \tag{1.3.2}$$

Here, for an isometry $A : \mathbb{R}^n \to \mathbb{R}^n$, the $2n$ multilinear functional E_A is defined by,

$$E_A(f_1, \ldots, f_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} f_k((Ax)_k) \overline{f}_{n+k}(x_k) dx, \tag{1.3.3}$$

where $x \in \mathbb{R}^n$, $x_k = \langle x, e_k \rangle$ and $(Ax)_k = \langle Ax, e_k \rangle$. (In the introduction, and in formulas (1.3.2), n is set to 3, but the work in this section is for arbitrary n.) As stated in the introduction, it turns out that one can gain significant insight into the dynamics of the systems associated to the Hamiltonians $\mathcal{H}_6$ and $\mathcal{H}_4$ by understanding properties of the functionals E_A. This section, therefore, is a general study of this family of functionals. We will examine the symmetries of E_A, its boundedness in L^2 and higher Sobolev spaces, and its relationship to the Hermite functions. Our motivation throughout is to relate these findings back to the dynamics of the Hamiltonian systems defined by $\mathcal{H}_6$ and $\mathcal{H}_4$. For this reason we will also present a number of results that relate properties of a generic multilinear functional $\mathcal{E}(f_1, \ldots, f_{2n})$, to the flow induced by the Hamiltonian $\mathcal{H}(f) = \mathcal{E}(f, \ldots, f)$ associated to it.

Our approach here is abstract, but we consider the abstraction justified for three reasons. First, it is efficient. Once we have proved, for example, L^2 local wellposedness for the partial differential equation induced by (1.3.3), it will immediately imply L^2 local wellposedness for the two *distinct* systems defined by (1.3.2). Second, our approach clarifies which structure in the Hamiltonians (1.3.2) is responsible for certain dynamics. As a byproduct, it suggests that many of the remarkable properties of these Hamiltonian systems (large number of symmetries, wellposedness in many spaces, Hermite functions as stationary waves) are generic. Third, one might expect that there are other Hamiltonian systems of mathematical or physical interest that can be cast into the framework suggested by the representations in (1.3.2). Our results here would immediately give significant insight into the dynamics of such systems.

1.3.1 · THE MULTILINEAR FUNCTIONAL FRAMEWORK

In what follows, plain Latin letters such as E_A, T_A and H_A will denote the specific multilinear functional defined by (1.3.3) and objects associated to it. Curly letters $\mathcal{E}$, $\mathcal{T}$, and $\mathcal{H}$ will denote a generic multilinear functional, multilinear operator and Hamiltonian respectively. All multilinear functionals take $2n$ arguments, are linear in the first n arguments and conjugate linear in the last n arguments, as in (1.3.3).

The functional properties of E_A depend strongly on the matrix properties of A. In the representations in (1.3.2), the matrices involved are all isometries, and we will find that this structural property plays a key role in the analysis. We will therefore assume throughout that the matrix A is an isometry.

Definition 1.3.1. (i) To each matrix A we associate a multilinear operator T_A defined implicitly by the formula,

$$\langle T_A(f_1,\dots,f_{2n-1}),g\rangle_{L^2} = 2\sum_{k=1}^{n} E_A(f_1,\dots,f_{n+k-1},g,f_{n+k},\dots,f_{2n-1}). \tag{1.3.4}$$

(ii) To each matrix A we associate a function H_A defined by $H_A(f) = E_A(f,\dots,f)$.

These definitions are motivated by the following theorem.

Theorem 1.3.1. *Suppose that a multilinear functional $\mathcal{E}(f_1,\dots,f_{2n})$ has the permutation symmetry,*

$$\mathcal{E}(f_1,\dots,f_n,f_{n+1},\dots,f_{2n}) = \overline{\mathcal{E}(f_{n+1},\dots,f_{2n},f_1,\dots,f_n)}. \tag{1.3.5}$$

Then $\mathcal{H}(f) = \mathcal{E}(f,\dots,f)$ is a real valued function and hence a Hamiltonian on the phase space $L^2(\mathbb{R}\to\mathbb{C})$. Hamilton's equation of motion is given by $i u_t(t) = \mathcal{T}(u(t),\dots,u(t))$ where the multilinear operator $\mathcal{T}$ is defined implicitely by

$$\langle \mathcal{T}(f_1,\dots,f_{2n-1}),g\rangle_{L^2} = 2\sum_{k=1}^{n} \mathcal{E}(f_1,\dots,f_{n+k-1},g,f_{n+k},\dots,f_{2n-1}). \tag{1.3.6}$$

Proof. First, if (1.3.5) is satisfied, then $\mathcal{H}(f) = \mathcal{E}(f,\dots,f) = \overline{\mathcal{E}(f,\dots,f)} = \overline{\mathcal{H}(f)}$, and therefore $\mathcal{H}(f)\in\mathbb{R}$.

In order to find Hamilton's equation of motion corresponding to $\mathcal{H}$, we first recall the Hamiltonian phase space structure of $L^2(\mathbb{R}\to\mathbb{C})$. A symplectic form on L^2 is given by $\omega(f,g) = -\mathrm{Im}\,\langle f,g\rangle_{L^2}$. Given a Hamiltonian $\mathcal{H}: L^2\to\mathbb{R}$, the symplectic gradient $\nabla_\omega\mathcal{H}$ is defined as the unique solution of the equation

$$\omega(\nabla_\omega\mathcal{H}(f),g) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\mathcal{H}(f+\epsilon g). \tag{1.3.7}$$

Hamilton's equation is then $u_t = \nabla_\omega\mathcal{H}(u)$.

In the case when $\mathcal{H}(f) = \mathcal{E}(f,\dots,f)$, we have, by multilinearity,

$$\begin{aligned}\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\mathcal{H}(f+\epsilon g) &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\mathcal{E}(f+\epsilon g,\dots,f+\epsilon g)\\ &= \sum_{k=1}^{2n}\mathcal{E}(\underbrace{f,\dots,f}_{k-1\text{ times}},g,\underbrace{f,\dots,f}_{2n-k\text{ times}}) = 2\mathrm{Re}\sum_{k=1}^{n}\mathcal{E}(\underbrace{f,\dots,f}_{n+k-1\text{ times}},g,\underbrace{f,\dots,f}_{2n-k\text{ times}}),\end{aligned} \tag{1.3.8}$$

where in the last step we used the permutation symmetry (1.3.5). On the other hand, setting,

$$i\nabla_w\mathcal{H}(f) = \mathcal{T}(f,\dots,f),$$

we find,

$$\omega(\nabla_\omega\mathcal{H}(f),g) = -\mathrm{Im}\,\langle i\mathcal{T}(f,\dots,f),g\rangle = \mathrm{Re}\,\langle\mathcal{T}(f,\dots,f),g\rangle. \tag{1.3.9}$$

By the definition of the symplectic gradient in (1.3.7), the right hand sides of (1.3.8) and (1.3.9) must match for all f and g. By replacing g by ig and using conjugate linearity, we see that this equality condition holding for all g actually implies that,

$$\langle \mathcal{T}(f,\dots,f), g\rangle = 2\sum_{k=1}^{n} \mathcal{E}(\underbrace{f,\dots,f}_{n+k-1 \text{ times}}, g, \underbrace{f,\dots,f}_{2n-k \text{ times}}),$$

which, in polarized from, is precisely (1.3.6).

Finally, Hamilton's equation is $iu_t = i\nabla_w H(u) = \mathcal{T}(u,\dots,u)$. □

For a generic isometry A, the functional E_A does not satisfy the permutation symmetry condition (1.3.5). However, if we define, for example,

$$\widetilde{E}_A(f_1,\dots,f_{2n}) = \frac{1}{2}\Big[E_A(f_1,\dots,f_{2n}) + \overline{E_A(f_{n+1},\dots,f_{2n},f_1,\dots,f_n)}\Big],$$

then $\widetilde{E}_A$ does satisfy (1.3.5), and all the properties of E_A we prove below carry over to $\widetilde{E}_A$. We will not be concerned with this point, because while the functionals E_A do not have the permutation symmetry (1.3.5), the functionals $\mathcal{E}_6$ and $\mathcal{E}_4$ defined in (1.1.8) and (1.1.10) do.

Before presenting general results on E_A, we give two concrete examples. These two examples illustrate how different isometries A can give rise to very different partial differential equations $iu_t = T_A(u,\dots,u)$.

Example 1. Take $n = 2$ and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity matrix. Then,

$$E_A(f_1,f_2,f_3,f_4) = \int_{\mathbb{R}^2} f_1(x_1)f_2(x_2)\overline{f_3(x_1)f_4(x_2)}dx = \langle f_1,f_3\rangle_{L^2}\langle f_2,f_4\rangle_{L^2},$$

and hence $H_A(f) = \|f\|_{L^2}^4$. We calculate,

$$\begin{aligned} T_A(f_1,f_2,f_3)(y) &= 2\left[\int_{\mathbb{R}} f_1(x_1)\left[f_2(x_2)\overline{f_3(x_2)}\right]\delta_{y=x_1}dx + \int_{\mathbb{R}}\left[f_1(x_1)\overline{f_3(x_1)}\right]f_2(x_2)\delta_{y=x_2}dx\right] \\ &= 2f_1(y)\langle f_2,f_3\rangle_{L^2} + 2f_2(y)\langle f_1,f_3\rangle_{L^2}. \end{aligned}$$

Hamilton's equation is then $iu_t = T_A(u,u,u) = 4u\|u\|_{L^2}^2$, which has a unique solution for initial data $u_0 \in L^2$ given by $u(x,t) = e^{4i\|u_0\|_{L^2}^2 t}u_0(x)$.

Example 2. Take $n = 2$ again, and let $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ be the rotation of $\mathbb{R}^2$ by $\pi/4$ radians. Then,

$$E_A(f_1,f_2,f_3,f_4) = \int_{\mathbb{R}^2} f_1\left(\frac{x_1-x_2}{\sqrt{2}}\right) f_2\left(\frac{x_1+x_2}{\sqrt{2}}\right)\overline{f_3(x_1)f_4(x_2)}dx,$$

and,

$$\begin{aligned} &T_A(f_1,f_2,f_3)(y) \\ &\quad = 2\left[\int_{\mathbb{R}} f_1\left(\frac{y-s}{\sqrt{2}}\right)f_2\left(\frac{y+s}{\sqrt{2}}\right)\overline{f_3(s)}ds + \int_{\mathbb{R}} f_1\left(\frac{s-y}{\sqrt{2}}\right)f_2\left(\frac{s+y}{\sqrt{2}}\right)\overline{f_3(s)}ds\right]. \end{aligned} \tag{1.3.10}$$

In this case it is not clear that the general solution of the equation $iu_t = T_A(u,u,u)$ can be written explicitly. However it is still possible to determine many properties of the flow. For example, one may verify by substitution that, for any $\alpha > 0$, the functions $u(x,t) = e^{i\sqrt{8\pi/\alpha}t}e^{-\alpha x^2}$ and $u(x,t) = xe^{-\alpha x^2}$, are explicit solutions of $iu_t = T_A(u,u,u)$ (the second solution does not depend on time). These solutions were both produced using Corollary 1.3.10 below.

1.3.2 · Symmetries of the functional and associated conservation laws

In this subsection we uncover some of the rich symmetry structure of the functional E_A. We recall that A is assumed to be an isometry throughout.

Theorem 1.3.2. *The functional E_A is invariant under the Fourier transform, that is,*

$$E_A(\widehat{f}_1,\dots,\widehat{f}_{2n}) = E_A(f_1,\dots,f_{2n}). \tag{1.3.11}$$

It follows that $T_A(\widehat{f}_1,\dots,\widehat{f}_{2n-1}))(\xi) = \widehat{T_A}(f_1,\dots,f_{2n-1})(\xi)$.

Proof. Because A is an isometry, we have $\langle \xi, Ax\rangle = \langle A^{-1}\xi, x\rangle$ for all $\xi, x \in \mathbb{R}^n$. Now calculating,

$$\begin{aligned}
E_A(\widehat{f}_1,\dots,\widehat{f}_{2n}) &= \int_{\mathbb{R}^n}\prod_{k=1}^n \widehat{f}_k((Ax)_k)\overline{\widehat{f}_{n+k}(x_k)}dx \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\prod_{k=1}^n\left(\int_{\mathbb{R}} e^{-i\xi_k(Ax)_k} f_k(\xi_k)d\xi_k\right)\left(\int_{\mathbb{R}} e^{i\nu_k x_k}\overline{f}_{n+k}(\nu_k)d\nu_k\right)dx \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{2n}} e^{-i\langle A^{-1}\xi-\nu,x\rangle}\prod_{k=1}^n f_k(\xi_k)\overline{f}_{n+k}(\nu_k)d\xi d\nu dx,
\end{aligned}$$

where in the last line we have used $\langle \xi, Ax\rangle + \langle \nu, x\rangle = \langle A^{-1}\xi - \nu, x\rangle$. We first change variables $y(\xi) = A^{-1}\xi - \nu$, or $\xi(y) = Ay + A\nu$. The determinant of this change of variables is 1 because A is an isometry. Performing the change of variables then gives the required identity,

$$\begin{aligned}
E_A(\widehat{f}_1,\dots,\widehat{f}_{2n}) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{2n}} e^{-i\langle y,x\rangle}\prod_{k=1}^n f_k(A\nu_k + Ay)\overline{f}_{n+k}(\nu_k)dy d\nu dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2n}}\prod_{k=1}^n f_k((A\nu)_k)\overline{f}_{n+k}(\nu_k)d\nu = E_A(f_1,\dots,f_{2n}),
\end{aligned}$$

where in the second equality we used the Fourier inversion identity (1.1.18) with $a = 1$.

For the operator statement, let g be an arbitrary element of L^2. Using the definition of T_A in (1.3.4) we have,

$$\begin{aligned}
\langle \widehat{T_A}(f_1,\dots,f_{2n-1}), \widehat{g}\rangle = \langle T_A(f_1,\dots,f_{2n-1}), g\rangle &= 2\sum_{k=1}^n E_A(f_1,\dots,f_{n+k-1},g,f_{n+k},\dots,f_{2n-1}) \\
&= 2\sum_{k=1}^n E_A(\widehat{f}_1,\dots,\widehat{f}_{n+k-1},\hat{g},\widehat{f}_{n+k},\dots,\widehat{f}_{2n-1}) \\
&= \langle T_A(\widehat{f}_1,\dots,\widehat{f}_{2n-1}), \widehat{g}\rangle.
\end{aligned} \tag{1.3.12}$$

The operator identity follows. □

Theorem 1.3.3. *The functional E_A is invariant under the following actions (for any λ):*

(i) Modulation: $f_k \mapsto e^{i\lambda} f_k$.

(ii) L^2 *scaling:* $f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x)$.

(iii) Quadratic modulation: $f_k \mapsto e^{i\lambda|x|^2} f_k$.

(iv) Schrödinger group: $f_k \mapsto e^{i\lambda\Delta} f_k$.

(v) Schrödinger with harmonic trapping group: $f_k \mapsto e^{i\lambda H} f_k$*, where* $H = -\Delta + |x|^2$.

If, in addition, A *satisfies* $Ae = e$*, where* $e = (1, \dots, 1) \in \mathbb{R}^n$*, then* E_A *is invariant under the following actions (for any* λ*):*

(vi) Linear modulation: $f_k \mapsto e^{i\lambda x} f_k$.

(vii) Translation: $f_k \mapsto f_k(\cdot + \lambda)$.

Proof. (i) We have,

$$E_A(e^{i\lambda} f_1, \dots, e^{i\lambda} f_{2n}) = \int_{\mathbb{R}^n} \prod_{k=1}^n e^{i\lambda} f_1((Ax)_k) e^{-i\lambda} \overline{f}_{2n}(x_k) dx = E_A(f_1, \dots, f_{2n}).$$

(ii) Let $f_k^\lambda(x) = \lambda^{1/2} f_k(\lambda x)$. We write out E_A and perform the change of variables $y = \lambda x$ (with $dy = \lambda^n dx$) to find,

$$\begin{aligned} E_A(f_1^\lambda, \dots, f_{2n}^\lambda) &= \lambda^n \int_{\mathbb{R}^n} \prod_{k=1}^n f_k(\lambda (Ax)_k) \overline{f}_{n+k}(\lambda x_k) dx \\ &= \int_{\mathbb{R}^n} \prod_{k=1}^n f((Ay)_k) \overline{f}(x_k) dx = E_A(f_1, \dots, f_{2n}). \end{aligned}$$

(iii) Because A is an isometry, $|Ax|^2 = |x|^2$ for all $x \in \mathbb{R}^n$. Using this, we have,

$$\begin{aligned} E_A(e^{i\lambda|x|^2} f_1, \dots, e^{i\lambda|x|^2} f_{2n}) &= \int_{\mathbb{R}^n} \prod_{k=1}^n e^{i\lambda|(Ax)_k|^2} f_k((Ax)_k) e^{-i\lambda|x_k|^2} \overline{f}_{n+k}(x_k) dx \\ &= \int_{\mathbb{R}^n} e^{i\lambda|Ax|^2} e^{-i\lambda|x|^2} \prod_{k=1}^n f_k((Ax)_k) \overline{f}_{n+k}(x_k) dx \\ &= E_A(f_1, \dots, f_{2n}). \end{aligned}$$

(iv) Using the previous part and the invariance of the Hamiltonian under the Fourier transform, we find,

$$\begin{aligned} E_A(e^{i\lambda\Delta} f_1, \dots, e^{i\lambda\Delta} f_{2n}) &= E_A(e^{-i\lambda|x|^2} \widehat{f}_1, \dots, e^{-i\lambda|x|^2} \widehat{f}_{2n}) \\ &= E_A(\widehat{f}_1, \dots, \widehat{f}_{2n}) = E_A(f_1, \dots, f_{2n}). \end{aligned}$$

(v) In this part we use t instead of λ, and show invariance of the functional under e^{itH}.

First, we note that if n is an integer then $e^{i(\pi/2 + n\pi)H} f = \widehat{f}$ (from, for instance, the Mehler formula (1.2.2)). The $t = \pi/2 + n\pi$ case thus follows from Theorem 1.3.2.

If $t \neq \pi/2+n\pi$ then we may represent $e^{itH} f$ using the *lens transform* [44]. This transform relates solutions of the free linear Schrödinger to the linear Schrödinger equation with harmonic trapping. Precisely, there holds,

$$(e^{itH} f_k)(x) = \frac{1}{\sqrt{\cos(2t)}}(e^{i(\tan(2t)/2)\Delta} f_k)\left(\frac{x}{\cos(2t)}\right) e^{ix^2 \tan(2t)/2}. \tag{1.3.13}$$

We substitute this expression into the functional. Using in turn the symmetries (iii) (with $\lambda = \tan(2t)/2$), (ii) (with $\lambda = 1/\cos(2t)$), and (iv) (with $\lambda = \tan(2t)/2$), we determine that,

$$\begin{aligned}
&E_A(e^{itH} f_1, \ldots, e^{itH} f_{2n}) \\
&\quad = E_A\left(\frac{1}{\sqrt{\cos(2t)}}(e^{i(\tan(2t)/2)\Delta} f_1)\left(\frac{x}{\cos(2t)}\right),\right. \\
&\qquad\qquad \left.\ldots, \frac{1}{\sqrt{\cos(2t)}}(e^{i(\tan(2t)/2)\Delta} f_{2n})\left(\frac{x}{\cos(2t)}\right)\right) \\
&\quad = E_A\left((e^{i(\tan(2t)/2)\Delta} f_1)(x), \ldots, (e^{i(\tan(2t)/2)\Delta} f_{2n})(x)\right) = E_A(f_1, \ldots, f_{2n}).
\end{aligned}$$

(vi) In these last two parts we assume that, in addition to being an isometry, the matrix A also satisfies $Ae = e$ for $e = (1, \ldots, 1) \in \mathbb{R}^n$. We then have,

$$\begin{aligned}
E_A(e^{i\lambda x} f_1, \ldots, e^{i\lambda x} f_{2n}) &= \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{i\lambda (Ax)_k} f_k((Ax)_k) e^{-i\lambda x_k} \overline{f}_{n+k}(x_k) dx \\
&= \int_{\mathbb{R}^n} e^{i\lambda\langle Ax,e\rangle} e^{-\lambda\langle x,e\rangle} \prod_{k=1}^{n} f_k((Ax)_k) \overline{f}_{n+k}(x_k) dx \\
&= E_A(f_1, \ldots, f_{2n}).
\end{aligned}$$

where in the last step we used $\langle Ax, e\rangle = \langle x, A^{-1}e\rangle = \langle x, e\rangle$.

(vii) This follows immediately from the previous part and the invariance of the functional under the Fourier transform, as in item (iv), noting that the Fourier transform takes $x \mapsto e^{i\lambda x} f(x)$ to $\xi \mapsto \widehat{f}(\xi + \lambda)$.

□

The symmetries of the functional E_A lead directly to commutator identities for the operator T_A.

Corollary 1.3.4. *We have the following commuter identities,*

$$e^{i\lambda Q} T_A(f_1, \ldots, f_{2n-1}) = T_A(e^{i\lambda Q} f_1, \ldots, e^{i\lambda Q} f_{2n-1}) \tag{1.3.14}$$

$$\begin{aligned}
Q T_A(f_1, \ldots, f_{2n-1}) &= \sum_{k=1}^{n} T_A(f_1, \ldots, f_{k-1}, Q f_k, f_{k+1}, \ldots f_{2n-1}) \\
&\qquad - \sum_{k=n+1}^{2n-1} T_A(f_1, \ldots, f_{k-1}, Q f_k, f_{k+1}, \ldots f_{2n-1}),
\end{aligned} \tag{1.3.15}$$

where Q is any of the following operators.

1. *For a generic isometry A, $Q = 1$, $Q = x^2$, $Q = \Delta$ and $Q = H$.*

2. If in addition $Ae = e$, where $e = (1, \dots, 1)$, $Q = x$, $Q = id/dx$.

Proof. For each of the operators Q, the flow map $e^{i\lambda Q}$ is an isometry of L^2 for all λ, and,

$$E_A(e^{i\lambda Q} f_1, \dots, e^{i\lambda Q} f_{2n}) = E_A(f_1, \dots, f_{2n}),$$

from Theorem 1.3.3. For each $g \in L^2$, we thus have,

$$\begin{aligned}
&\langle e^{i\lambda Q} T_A(f_1, \dots, f_{2n-1}), g\rangle_{L^2} \\
&\quad = \langle T_A(f_1, \dots, f_{2n-1}), e^{-i\lambda Q} g\rangle_{L^2} \\
&\quad = 2\sum_{k=1}^{n} E_A\left(f_1, \dots, f_{n+k-1}, e^{-i\lambda Q} g, f_{n+k}, \dots, f_{2n-1},\right) \\
&\quad = 2\sum_{k=1}^{n} E_A\left(e^{i\lambda Q} f_1, \dots, e^{i\lambda Q} f_{n+k-1}, g, e^{i\lambda Q} f_{n+k}, \dots, e^{i\lambda Q} f_{2n-1},\right) \\
&\quad = \langle T_A(e^{i\lambda Q} f_1, \dots, e^{i\lambda Q} f_{2n-1}), g\rangle_{L^2},
\end{aligned}$$

which gives (1.3.14). To get (1.3.15), differentiate (1.3.14) with respect to λ and set $\lambda = 0$. □

In Hamiltonian mechanics, the primary purpose of finding symmetries is to determine conservation laws. These two concepts are linked through Noether's theorem. We have seen, in Theorem 1.3.1, that in the present context if a functional $\mathcal{E}$ satisfies the permutation symmetry (1.3.5), then the functional gives rise to a Hamiltonian $\mathcal{H}$ and Hamilton's equation of motion is $iu_t = \mathcal{T}(u, \dots, u)$. With this Hamiltonian structure, a version of Noether's Theorem applies.

Theorem 1.3.5 (Noether's Theorem)**.** *Let $\mathcal{E}$ be a multilinear functional that satisfies the permutation symmetry* (1.3.5)*. Suppose that Q is a self-adjoint operator on L^2 such that*

$$\mathcal{E}(e^{i\lambda Q} f_1, \dots, e^{i\lambda Q} f_{2n}) = \mathcal{E}(f_1, \dots, f_{2n}). \tag{1.3.16}$$

Then the quantity $\langle Qf, f\rangle_{L^2}$ is conserved by the Hamiltonian flow of $\mathcal{H}(f) = \mathcal{E}(f, \dots, f)$.

Proof. We first show that $\langle \mathcal{T}(f, \dots, f), Qf\rangle \in \mathbb{R}$. Differentiating equation (1.3.16) with respect to λ and setting $\lambda = 0$ gives,

$$\sum_{k=1}^{n} \mathcal{E}(\underbrace{f, \dots, f}_{k-1 \text{ times}}, Qf, \underbrace{f, \dots, f}_{2n-k \text{ times}}) = \sum_{k=1}^{n} \mathcal{E}(\underbrace{f, \dots, f}_{n+k-1 \text{ times}}, Qf, \underbrace{f, \dots, f}_{n-k \text{ times}}); \tag{1.3.17}$$

the sign being determined by the linearity or conjugate linearity of each component. Now using the permutation symmetry (1.3.5) followed by (1.3.17) gives,

$$\begin{aligned}
\overline{\langle \mathcal{T}(f, \dots, f), Qf\rangle} &= 2\sum_{k=1}^{n} \overline{\mathcal{E}(\underbrace{f, \dots, f}_{n+k-1 \text{ times}}, Qf, \underbrace{f, \dots, f}_{n-k \text{ times}})} = 2\sum_{k=1}^{n} \mathcal{E}(\underbrace{f, \dots, f}_{k-1 \text{ times}}, Qf, \underbrace{f, \dots, f}_{2n-k \text{ times}}) \\
&= 2\sum_{k=1}^{n} \mathcal{E}(\underbrace{f, \dots, f}_{n+k-1 \text{ times}}, Qf, \underbrace{f, \dots, f}_{n-k \text{ times}}) = \langle \mathcal{T}(f, \dots, f), Qf\rangle,
\end{aligned}$$

Symmetry $e^{i\lambda Q}$ of $\mathcal{E}$	Operator Q commuting with $\mathcal{T}$	Conserved quantity $\langle Qf, f\rangle$
$f \mapsto e^{i\lambda} f$	1	$\int_{\mathbb{R}} \lvert f(x)\rvert^2 dx$
$f \mapsto f_\lambda$		$\int_{\mathbb{R}} [ixf'(x) + f(x)]\,\overline{f}(x)dx$
$f \mapsto e^{i\lambda \lvert x\rvert^2} f$	x^2	$\int_{\mathbb{R}} \lvert xf(x)\rvert^2 dx$
$f \mapsto e^{i\lambda\Delta} f$	Δ	$\int_{\mathbb{R}} \lvert f'(x)\rvert^2 dx$
$f \mapsto e^{i\lambda\mathcal{H}} f$	H	$\int_{\mathbb{R}} \lvert xf(x)\rvert^2 + \lvert f'(x)\rvert^2 dx$
$f \mapsto e^{i\lambda x} f$	x	$\int_{\mathbb{R}} x\lvert f(x)\rvert^2 dx$
$f \mapsto f(\cdot + \lambda)$	id/dx	$\int_{\mathbb{R}} if'(x)\overline{f}(x)dx$

Table 1.1: Relationship between the symmetries of $\mathcal{E}$ and the conserved quantites of the Hamiltonian flow, as given by Noether's Theorem.

which shows that $\langle \mathcal{T}(f, \ldots, f), Qf\rangle$ is real. Then, if $if_t = \mathcal{T}(f, \ldots, f)$, we have,

$$\begin{aligned}\frac{d}{dt}\langle Qf, f\rangle &= i\,[\langle Q\mathcal{T}(f, \ldots, f), f\rangle - \langle Qf, \mathcal{T}(f, \ldots, f)\rangle] \\ &= i\,[\langle \mathcal{T}(f, \ldots, f), Qf\rangle - \langle Qf, \mathcal{T}(f, \ldots, f)\rangle] = 2i\,\mathrm{Im}\,\langle \mathcal{T}(f, \ldots, f), Qf\rangle = 0,\end{aligned}$$

so $\langle Qf, f\rangle$ is constant. □

Table 1.1 summarizes the relationships between the symmetries of E_A described in Theorem 1.3.3, the associated commuting operators in Corrolary 1.3.4, and the conserved quantities given by Noether's Theorem. As discussed previously, the functional E_A does not automatically satisfy the permutation symmetry (1.3.5) so Noether's Theorem does not apply directly; however, the Hamiltonian systems defined by $\mathcal{H}_6$ and $\mathcal{H}_4$ do satisfy (1.3.5) and so will have a number of these conserved quantities as a consequence of symmetries induced by Theorem 1.3.3.

1.3.3 · Boundedness and wellposedness

In this subsection we establish an L^2 bound on E_A, and bounds of the form,

$$\|T_A(f_1, \ldots, f_{2n-1})\|_X \le C\,\|f_1\|_X \cdots \|f_{2n-1}\|_X, \tag{1.3.18}$$

for $X = L^2$, $X = L^{2,\sigma}$ and $X = H^\sigma$. We will then show how these bounds imply local existence of solutions to Hamilton's equation $iu_t = T_A(u, \ldots, u)$ in the space X. By employing some of the conservation laws derived in the last section, it is possible to establish global existence in L^2 and other spaces for certain functionals.

Theorem 1.3.6. *There holds the bound,*

$$|E_A(f_1, \ldots, f_{2n})| \le \prod_{k=1}^{2n} \|f_k\|_{L^2}. \tag{1.3.19}$$

In particular we have $H_A(f) \le \|f\|_{L^2}^{2n}$. Both of these bounds are sharp.

Proof. The proof involves one use of the Cauchy–Schwarz inequality. We have,

$$
\begin{aligned}
|E_A(f_1,\ldots,f_{2n})| &= \left| \int_{\mathbb{R}^n} \prod_{k=1}^n f_k((Ax)_k)\overline{f}_{n+k}(x_k)dx \right| \\
&\leq \left(\int_{\mathbb{R}^n} \prod_{k=1}^n |f_k((Ax)_k)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \prod_{k=1}^n |f_{n+k}(x_k)|^2 dx \right)^{1/2}.
\end{aligned}
$$

In the first integral we perform the change of variables $y = Ax$. Because A is an isometry, the determinant of this change of variables is 1, and so by Fubini's Theorem,

$$
|E_A(f_1,\ldots,f_{2n})| \leq \left(\int_{\mathbb{R}^n} \prod_{k=1}^n |f_k(y_k)|^2 dy \right)^{1/2} \left(\int_{\mathbb{R}^n} \prod_{k=1}^n |f_{n+k}(x_k)|^2 dx \right)^{1/2} = \prod_{k=1}^{2n} \|f_k\|_{L^2},
$$

which is the bound for E_A. Setting $f_k = f$ for all k gives the bound $H_A(f) \leq \|f\|_{L^2}^2$.

From the Cauchy–Schwarz inequality, we have the equality condition,

$$
\prod_{k=1}^n f_k((Ax)_k) = \prod_{k=1}^n f_{n+k}(x_k). \tag{1.3.20}
$$

By assumption, A is an isometry, so that $\sum_{k=1}^n |(Ax)_k|^2 = |Ax|^2 = |x|^2 = \sum_{k=1}^n |x_k|^2$. This immediately gives that the Gaussian $G(x) = e^{-\alpha x^2}$ satisfies the equality condition (1.3.20), and hence that $H(G) = \|G\|_{L^2}^{2n}$. The inequality (1.3.19) is thus sharp. □

In a later subsection we discuss the classification of all functions $\{f_k\}_{k=1}^{2n}$ that saturate the multilinear functional inequality (1.3.19). The two examples on page 24 show that, in general, such a classification will depend on the matrix properties of A. In Example 1, $H_A(f) = \|f\|_{L^2}^4$, and so the bound $H_A(f) \leq \|f\|_{L^2}^4$ in Theorem 1.3.6 is *always* equality. This is not the case in Example 2. In Theorem 1.3.11, we will find that if A is not a signed permutation matrix – namely that there is at least one basis element e_k such that Ae_k is a linear sum of at least two other e_j basis elements – then the equality $H_A(f) = \|f\|_{L^2}^{2k}$ holds only if f is a Gaussian.

In the meantime, we use the inequality (1.3.19) to establish bounds for the multilinear operator T_A.

Corollary 1.3.7. *There holds the bound,*

$$
\|T_A(f_1,\ldots,f_{2n-1})\|_X \leq C_n \prod_{k=1}^{2n-1} \|f_k\|_X^{2n-1},
$$

for the following spaces.

(i) $X = L^2$ *with* $C_n = 2n$;

(ii) $X = L^{2,\sigma}$, *for any* $\sigma \geq 0$ *with* $C_n = 2n^{2+\sigma}$.

(iii) $X = H^\sigma$, *for any* $\sigma \geq 0$ *with* $C_n = 2n^{2+\sigma}$.

It is important that the boundedness constant C_n is independent of A. This implies that if we have a composite Hamiltonian of the form,

$$
\int_\Omega \phi(\lambda) H_{A(\lambda)}(f) d\lambda,
$$

then the associated operator will be bounded once ϕ is integrable. This is precisely how Corollary 1.3.7 will be applied to the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$.

Proof. (i) We argue by duality using the implicit representation (1.3.4). By the bound on E_A from Theorem 1.3.6 we have,

$$
\begin{aligned}
|\langle T_A(f_1,\dots,f_{2n-1}),g\rangle_{L^2}| &\le 2\sum_{k=n+1}^{2n}|E_A(f_1,\dots,f_{n+m-1},g,f_{n+m},\dots,f_{2n-1})|\\
&\le 2\sum_{k=n+1}^{2n}\prod_{k=1}^{2n-1}\|f_k\|_{L^2}\|g\|_{L^2} = \left(2n\prod_{k=1}^{2n-1}\|f_k\|_{L^2}\right)\|g\|_{L^2}.
\end{aligned}
$$

This gives the result for $X = L^2$.

(ii) Fix $x\in\mathbb{R}^n$. Because A is an isometry we have, for every m, $|x_m|^2 \le |x|^2 = |Ax|^2 = \sum_{k=1}^n |(Ax)_k|^2$. Therefore, for fixed m, there is an integer l such that $|x_m|^2 \le n|(Ax)_l|^2$. With $\langle\rangle$ denoting the Japanese bracket, we then have $\langle x_m\rangle \le n\langle (Ax)_l\rangle$ and so,

$$
\langle x_m\rangle \le n\left(\prod_{k=1,k\neq m}^{n}\langle (Ax)_k\rangle\rangle\langle x_k\rangle\right)\langle (Ax)_m\rangle,
$$

because in all cases $\langle t\rangle \ge 1$. In terms of the functional E_A, this gives,

$$
\begin{aligned}
&E_A(|f_1|,\dots,|f_{k-1}|,\langle t\rangle^\sigma|f_k|,|f_{k-1}|,\dots,|f_{2n}|)\\
&\qquad\le n^\sigma E_A(\langle t\rangle^\sigma|f_1|,\dots,\langle t\rangle^\sigma|f_{k-1}|,|f_k|,\langle t\rangle^\sigma|f_{k-1}|,\dots,\langle t\rangle^\sigma|f_{2n}|).
\end{aligned}
\tag{1.3.21}
$$

Now applying this to T_A, we have,

$$
\begin{aligned}
&\langle T_A(f_1,\dots,f_{2n-1}),g\rangle_{L^{2,\sigma}}\\
&\quad= \langle T_A(f_1,\dots,f_{2n-1}),\langle t\rangle^{2\sigma}g\rangle_{L^2}\\
&\quad= 2n\sum_{k=n+1}^{2n}E_A(f_1,\dots,f_{k-1},\langle t\rangle^{2\sigma}g,f_k,\dots,f_{2n-1})\\
&\quad\le 2n^{1+\sigma}\sum_{k=n+1}^{2n}E_A(\langle t\rangle^\sigma|f_1|,\dots,\langle t\rangle^\sigma|f_{k-1}|,\langle t\rangle^\sigma|g|,\langle t\rangle^\sigma|f_k|,\dots,\langle t\rangle^\sigma|f_{2n-1}|)\\
&\quad\le\left(2n^{2+\sigma}\prod_{k=1}^{2n-1}\|f_k\|_{L^{2,\sigma}}\right)\|g\|_{L^{2,\sigma}},
\end{aligned}
$$

which gives the result for $X = L^{2,\sigma}$.

(iii) This follows from (ii) using the invariance of the operator T_A under the Fourier transform. □

The next proposition shows that, in general, bounds of the form determined in the previous proposition imply local wellposedness.

Proposition 1.3.8. *Suppose that $\mathcal{T} : X^m \to X$ is a multilinear operator that is bounded on a Banach space X; that* *is,*

$$\|\mathcal{T}(f_1,\dots,f_m)\|_X \le C_{\mathcal{T}} \prod_{k=1}^{m} \|f_k\|_X$$

for some $C_{\mathcal{T}}$. Set $\mathcal{T}(f) = \mathcal{T}(f,\dots,f)$. For every $f_0 \in X$ there is a $T > 0$ and unique local solution to the Cauchy problem for Hamiltonian's equation,

$$\begin{aligned} i f_t &= \mathcal{T}(f), \\ f(t=0) &= f_0, \end{aligned} \tag{1.3.22}$$

in the space $L^\infty([0,T],X)$. The time of existence T depends only on $\|f_0\|_X$.

Proof. The Duhamel formulation of the Cauchy problem (1.3.22) is

$$f(t) := R(f(t)) = f_0 + \int_0^t \mathcal{T}(f(s))ds.$$

To prove local existence we will show that there is a $T = T(\|f_0\|_X)$ such that the operator R is a contraction on $Y_T = L^\infty([0,T],X)$. The proposition then follows from Banach's Fixed Point Theorem.

We first have the bound,

$$\|R(f(t))\|_X \le \|f_0\|_X + \int_0^t \|\mathcal{T}(f(s))\|_X ds \le \|f_0\|_X + C_{\mathcal{T}} \int_0^t \|f(s)\|_X^m ds \le \|f_0\|_X + tC_{\mathcal{T}}\|f\|_{Y_t}^m. \tag{1.3.23}$$

Next we have $\|R(f_1(t)) - R(f_2(t))\|_X \le \int_0^t \|\mathcal{T}(f_1(s)) - \mathcal{T}(f_2(s))\|_X ds$. To evaluate the right hand side, we use multilinearity to write,

$$\mathcal{T}(f_1(s)) - \mathcal{T}(f_2(s)) = \mathcal{T}(f_1,\dots,f_1) - \mathcal{T}(f_2,\dots,f_2) = \sum_{k=1}^{m} \mathcal{T}(\underbrace{f_1,\dots,f_1}_{k-1 \text{ times}}, f_1 - f_2, \underbrace{f_2,\dots,f_2}_{m-k \text{ times}}).$$

This equality gives,

$$\begin{aligned} \|\mathcal{T}(f_1(s)) - \mathcal{T}(f_2(s))\|_X &\le C_{\mathcal{T}} \sum_{k=1}^{m-1} \|f_1\|_X^{k-1} \|f_1 - f_2\|_X \|f_2\|_X^{m-k} \\ &\le \|f_1 - f_2\|_X (m-1) C_{\mathcal{T}} \left[\|f_1\|_X^{m-1} + \|f_2\|_X^{m-1}\right], \end{aligned}$$

and then,

$$\|R(f_1(t)) - R(f_2(t))\|_{Y_t} \le \|f_1 - f_2\|_{Y_t} \left(t(m-1)C_{\mathcal{T}}\left[\|f_1\|_{Y_t}^{m-2} + \|f_2\|_{Y_t}^{m-2}\right]\right). \tag{1.3.24}$$

From (1.3.23) and (1.3.24) it follows that for any $M > \|f_0\|_X$ we may choose T such that $R : B_{Y_T}(0,M) \to B_{Y_T}(0,M)$ and R a contraction on this space. □

It is key in the proposition that the local time of existence depends only on $\|f_0\|_X$. This enables one to pair the local wellposedness result with a conservation law in the previous subsection to get global wellposedness. We will do precisely this to prove global wellposedness in L^2 of solutions to the Hamiltonian systems $\mathcal{H}_6$ and $\mathcal{H}_4$.

The symmetry of the functional under the action e^{itH} implies interesting explicit relationships between E_A, T_A and the Hermite functions $\{\phi_n\}_{n=0}^{\infty}$.

Theorem 1.3.9. *If $\sum_{k=1}^{n} m_k - \sum_{k=n+1}^{2n} m_k \neq 0$, then $E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}) = 0$. This implies that,*

$$T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}}) = C\phi_l, \tag{1.3.25}$$

for some constant C where $l = \sum_{k=1}^{n} m_k - \sum_{k=n+1}^{2n-1} m_k$. (We understand that if $l < 0$ then $\phi_l := 0$.)

Proof. By the symmetry under the action e^{itH}, we have,

$$\begin{aligned} E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}) &= E_A(e^{itH}\phi_{m_1}, \ldots, e^{itH}\phi_{m_{2n}}) \\ &= E_A(e^{it(2m_1+1)}\phi_{m_1}, \ldots, e^{it(2m_{2n}+1)}\phi_{m_{2n}}) \\ &= e^{it2L} E_A(\phi_{m_1}, \ldots, \phi_{m_{2n}}), \end{aligned}$$

where $L = \sum_{k=1}^{n} m_k - \sum_{k=n+1}^{2n} m_k$. If $L \neq 0$, as in the statement of the theorem, then necessarily the E_A term here is 0.

Next, let $m_1, \ldots, m_{2n-1}$ be given, and fix $p \neq \sum_{k=1}^{n} m_k - \sum_{k=n+1}^{2n-1} m_k$. We have,

$$\langle T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}}), \phi_p \rangle_{L^2} = 2 \sum_{k=1}^{n} E_A(\phi_{m_1}, \ldots, \phi_{m_{n+k-1}}, \phi_p, \phi_{m_{n+k}}, \ldots, \phi_{m_{2n}}) = 0,$$

by the previous part and because $p \neq \sum_{k=1}^{n} m_k - \sum_{k=n+1}^{2n-1} m_k$. The function $T_A(\phi_{m_1}, \ldots, \phi_{m_{2n-1}})$ is thus orthogonal to ϕ_p for all such p. Because the Hermite functions are an orthonormal basis of L^2, we must have (1.3.25). □

Corollary 1.3.10. *Set $\omega_n = 2nH_A(\phi_n)$. Then $u(t,x) = e^{it\omega_n}\phi_n(x)$ is a stationary wave solution of $iu_t = T_A(u, \ldots, u)$.*

Proof. From the previous proposition, we know that,

$$T_A(\phi_n, \ldots, \phi_n) = \omega_n \phi_n, \tag{1.3.26}$$

for some ω_n. It it follows that $e^{-it\omega_n}\phi_n(x)$ is a solution of $iu_t = T_A(u, \ldots, u)$.

Taking the inner product of both sides of (1.3.26) with ϕ_n and using (1.3.4), we find that $\omega_n = 2nH_A(\phi_n)$. □

By applying the symmetries of E_A given in Theorem 1.3.3 to ϕ_n, one can obtain other stationary wave solutions. In fact, with this procedure one finds that all functions of the form, $ae^{ibx^2}\phi_n(cx)$ are stationary waves, where $a \in \mathbb{C}$ and $b, c \in \mathbb{R}$. If in addition the matrix A satisfies $A(1, \ldots, 1) = (1, \ldots, 1)$, then all functions of the form, $ae^{ibx^2+idx}\phi_n(cx + e)$ are stationary waves, where $a \in \mathbb{C}$ and $b, c, d, e \in \mathbb{R}$.

We note that in higher dimensions, as in the work of [25], the dynamics of the eigenvectors of H under the continuous resonant equation are more interesting because the eigenspaces of H are multidimensional. This allows for more complicated dynamics than simply stationary waves. In dimension one, each eigenspace of H is one dimensional (being spanned by ϕ_n alone) and it necessarily follows that any eigenvector of H corresponds simply to a stationary wave solution.

We have previously established the inequality $|H_A(f)| \le \|f\|_{L^2}^{2n}$ and showed that it is sharp. In this section we discuss the classification those functions that saturate the inequality. While classifying the cases of equality in an inequality is often illuminating, in the theory of Hamiltonians on the phase space L^2 it is especially relevant. In all the Hamiltonian systems discussed in this chapter, the L^2 norm of f is conserved by the Hamiltonian flow, and the Hamiltonian $\mathcal{H}$ itself is always conserved. Therefore, if one takes as initial data to the flow a function that saturates the inequality $\mathcal{H}(f) \le \|f\|_{L^2}^{2n}$, for all fixed future times t the solution $x \mapsto f(x,t)$ will still saturate the inequality. The set of all saturating functions is thus closed under the flow. Identifying the set of saturating functions can then lead to interesting explicit solutions. For the Hamiltonians in this chapter, however, we will find that the saturating functions all correspond simply to stationary wave solutions.

The two examples on page 24 show that the set of saturating functions for the inequality $|H_A(f)| \le \|f\|_{L^2}$ will depend on the matrix properties of A. Example 1 shows that if A is the identity, or more generally a permutation, then the inequality is *always* equality. If A is a signed permutation, for example the negative of the identity, then there is equality if and only if f is even, as we will see. However if A is not a signed permutation, as in Example 2, then the set of saturating functions is much smaller.

Theorem 1.3.11. *Suppose that A is an isometry which is not a signed permutation – that is, there is some i such that Ae_i is the linear sum of at least two basis elements e_j. Let $e = (1, \dots, 1)$ denote the vector in $\mathbb{R}^n$ with all its entries 1.*

(i) If $Ae = e$, then we have the equality $H_A(f) = \|f\|_{L^2}^{2n}$ if and only if $f(x) = ce^{-\alpha x^2 + \beta x}$ for some $c, \alpha, \beta \in \mathbb{C}$ with $Re\,\alpha > 0$.

(ii) If $Ae \ne e$, then we have the equality $H_A(f) = \|f\|_{L^2}^{2n}$ if and only if $f(x) = ce^{-\alpha x^2}$ for some $c, \alpha \in \mathbb{C}$ with $Re\,\alpha > 0$.

Proof of the 'if' statements. To check the equality case for Gaussians $f(x) = ce^{-\alpha x^2 + \beta x}$, we substitute f into both sides of (1.3.20), which was the equality condition in our use of the Cauchy–Schwarz inequality. The left hand side is,

$$\prod_{k=1}^{n} f((Ax)_k) = c^n \prod_{k=1}^{n} \exp\left(-\alpha\langle Ax, e_k\rangle^2 + \beta\langle x, e_k\rangle\right) = c^n \exp\left(-\alpha \sum_{k=1}^{n} \langle Ax, e_k\rangle^2 + \beta\langle Ax, e_k\rangle\right) \quad (1.3.27)$$

$$= c^n \exp\left(-\alpha|Ax|^2 + \beta\langle Ax, e\rangle\right) = c^n \exp\left(-\alpha|Ax|^2 + \beta\langle x, A^{-1}e\rangle\right), \quad (1.3.28)$$

where in the last line we have used $\langle Ax, e\rangle = \langle x, A^{-1}e\rangle$ coming from A being an isometry.

An identical computation shows that the right hand side of (1.3.20) is,

$$\prod_{k=1}^{n} f(x_k) = c^n \exp\left(-\alpha|x|^2 + \beta\langle x, v\rangle\right).$$

If $Ae = e$, so that $e = A^{-1}e$, then both sides are equal for all c, α and β, which proves the 'if' part of item (i). On the other hand, if $Ae \ne e$ then there is an $x \in \mathbb{R}^n$ such that $\langle x, A^{-1}e\rangle \ne \langle x, e\rangle$. Hence for equality to hold we necessarily have $\beta = 0$. However with $\beta = 0$ both sides are equal, and so the 'if' part of item (ii) is proved. □

The 'only if' part of the theorem follows from a more general result that classifies those functions that saturate the multilinear functional inequality $|E_A(f_1, \dots, f_{2n})| \le \|f_1\|_{L^2} \cdots \|f_{2n}\|_{L^2}$. Our classification result, given in Theorem 1.3.12 below, is not new. The multilinear functional inequality (1.3.19) is a specific example of a geometric Brasscamp–Lieb inequality. The Brasscamp–Lieb inequalities originated in [10] as generalizations of the Hölder inequality. The special class of geometric Brasscamp–Lieb inequalities was introduced in [1], and it was subsequently shown, in [2], that they are maximized only for Gaussians. Theorem 1.3.12 is a special case of this broad result. Our proof of this special case is, it seems, new, and we think worthy of presentation; however, because the result is not new and not essential to other parts of this chapter, the proof is presented in Appendix A.

Theorem 1.3.12. *Let A be an isometry. Denote $\tilde{f}_k(x) = f_k(-x)$. There exists integers m and l, with $0 \leq m \leq l \leq n$, and two permutations σ_1 and σ_2 of the integers $\{1, \dots, n\}$ such that*

$$E_A(f_1, \dots, f_{2n}) = \prod_{k=1}^{m} \langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)} \rangle \prod_{k=m+1}^{l} \langle f_{\sigma_1(k)}, \tilde{f}_{n+\sigma_2(k)} \rangle E_B(f_{\sigma_1(l+1)}, \dots f_{\sigma_1(n)}, f_{n+\sigma_2(l+1)}, \dots f_{n+\sigma_2(n)}),$$

where the matrix $B : \mathbb{R}^{n-l} \to \mathbb{R}^{n-l}$ has no permutation part; that is, for all k and j, $Be_k \neq \pm e_j$.

We then have $|E_A(f_1, \dots, f_{2n})| = \|f_1\|_{L^2} \cdots \|f_{2n}\|_{L^2}$ if and only if the following three conditions hold.

1. *For $k = 1, \dots, m$, $f_{\sigma_1(k)}(x) = C_k f_{n+\sigma_2(k)}(x)$, for some $C_k \in \mathbb{R}$.*
2. *For $k = m+1, \dots, l$, $f_{\sigma_1(k)}(x) = C_k f_{n+\sigma_2(k)}(-x)$, for some $C_k \in \mathbb{R}$.*
3. *For $k = l+1, \dots, n$, $f_{\sigma_1(k)}(x)$ and $f_{n+\sigma_2(k)}(x)$ are Gaussians.*

Proof of the 'only if' statements in Theorem 1.3.11, assuming Theorem 1.3.12. Because A is not a signed permutation, we must have $l < n$ in the notation of Theorem 1.3.12. Then in, Theorem 1.3.12, condition 3 applies. Hence to have the equality $H_A(f) = E_A(f, \dots, f) = \|f\|_{L^2}^{2n}$, f must necessarily be a Gaussian. □

1.3.6 · Regularity of stationary waves: a lemma

In this final part of our general study of the functional E_A, we prove a weight transfer lemma that is similar to formula (1.3.21). This lemma will be crucial later to proving that stationary waves of the Hamiltonian systems (1.1.7) and (1.1.9) equation are analytic and exponentially decaying once they are in L^2.

For fixed $\mu, \epsilon > 0$, define,

$$G_{\mu,\epsilon}(x) = \exp\left(\frac{\mu x^2}{1 + \epsilon x^2}\right).$$

Lemma 1.3.13. *If $\{f_k\}_{k=1}^{2n}$ are positive functions, then,*

$$E_A(f_1, \dots, f_{2n-1}, f_{2n} G_{\mu,\epsilon}) \leq E_A(f_1 G_{\mu,\epsilon}, \dots, f_{2n-1} G_{\mu,\epsilon}, f_{2n}).$$

Proof. Define $F_{\mu,\epsilon} = \mu|x|/(1 + \epsilon|x|)$. We clearly have $G_{\mu,\epsilon}(x) = \exp(F_{\mu,\epsilon}(x^2))$.

We record two properties of $F_{\mu,\epsilon}$. First, for $x > 0$, $F_{\mu,\epsilon}$ is increasing. This may be seen from,

$$\frac{d}{dx}\left(\mu \frac{x}{1 + \epsilon x}\right) = \mu \frac{1(1 + \epsilon x) - x(\epsilon)}{(1 + \epsilon x)^2} = \mu \frac{1}{(1 + \epsilon x)^2} > 0.$$

Next, we have $F_{\mu,\epsilon}(x_1 + x_2) \leq F_{\mu,\epsilon}(x_1) + F_{\mu,\epsilon}(x_2)$. This may be seen from,

$$F_{\mu,\epsilon}(x_1 + x_2) = F_{\mu,\epsilon}(|x_1 + x_2|) \leq F_{\mu,\epsilon}(|x_1| + |x_2|) = \mu \frac{|x_1| + |x_2|}{1 + \epsilon|x_1| + \epsilon|x_2|}$$
$$\leq \mu \frac{|x_1|}{1 + \epsilon|x_1|} + \mu \frac{|x_2|}{1 + \epsilon|x_2|} = F_{\mu,\epsilon}(x_1) + F_{\mu,\epsilon}(x_2).$$

Now because A is an isometry we have, for all $x \in \mathbb{R}^n$, $x_n^2 = \sum_{k=1}^{n}(Ax)_k^2 - \sum_{k=1}^{n-1}(x_k)^2$ and hence by the sublinearity property of $F_{\mu,\epsilon}$,

$$F_{\mu,\epsilon}(x_n^2) = F_{\mu,\epsilon}\left(\sum_{k=1}^{n}(Ax)_k^2 + \sum_{k=1}^{n-1} -(x_k)^2\right) \leq \sum_{k=1}^{n} F_{\mu,\epsilon}((Ax)_k^2) + \sum_{k=1}^{n-1} F_{\mu,\epsilon}(x_k^2),$$

Then, because $x \mapsto e^x$ is increasing,

$$G_{\mu,\epsilon}(x_n) = \exp(F_{\mu,\epsilon}(x_n^2)) \leq \prod_{k=1}^{n} G_{\mu,\epsilon}((Ax)_k) \prod_{k=1}^{n-1} G_{\mu,\epsilon}(x_k).$$

Applying this to E_A, we have,

$$E_A(f_1, \dots, f_{2n-1}, f_{2n}G_{\mu,\epsilon}) = \int_{\mathbb{R}^n} \left(\prod_{k=1}^{n-1} f_k((Ax)_k)\overline{f}_{n+k}(x_k)\right) f_n((Ax)_n)\overline{f}_{2n}(x_n)G_{\mu,\epsilon}(x_n)dx$$
$$\leq \int_{\mathbb{R}^n} \left(\prod_{k=1}^{n-1} f_k((Ax)_k)G_{\mu,\epsilon}((Ax)_k)\overline{f}_{n+k}(x_k)G_{\mu,\epsilon}(x_k)\right)$$
$$f_n((Ax)_n)G_{\mu,\epsilon}((Ax)_n)\overline{f}_{2n}(x_n)dx$$
$$= E_A(f_1G_{\mu,\epsilon}, \dots, f_{2n-1}G_{\mu,\epsilon}, f_{2n}),$$

which is what we wanted to prove. □

§1.4 · THE QUINTIC RESONANT EQUATION

We now turn to the first resonant Hamiltonian,

$$\mathcal{H}_6(f) = \frac{2}{\pi}\|e^{itH} f\|_{L_t^6 L_x^6} = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\int_{\mathbb{R}} |e^{itH} f(x)|^6 dxdt, \tag{1.4.1}$$

which has a corresponding multilinear functional,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\int_{\mathbb{R}} (e^{itH} f_1)(e^{itH} f_2)(e^{itH} f_3)\overline{(e^{itH} f_4)(e^{itH} f_5)(e^{itH} f_6)}dxdt. \tag{1.4.2}$$

We point out right away that, in contrast to the multilinear functionals E_A studied in the previous section, the functional $\mathcal{E}_6$ has a large number of permutation symmetries. For any two permutations of three elements

$\sigma, \sigma' \in S_3$, we have,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \mathcal{E}_6(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}, f_{\sigma'(4)}, f_{\sigma'(5)}, f_{\sigma'(6)}), \tag{1.4.3}$$

as well as the standard symmetry,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \overline{\mathcal{E}_6(f_4, f_5, f_6, f_1, f_2, f_3)}. \tag{1.4.4}$$

These symmetries simplify the formulas for Hamilton's equation. In fact, by Theorem 1.3.1, $\mathcal{H}_6$ defines a Hamiltonian flow on the phase space L^2 with Hamilton's equation given by

$$iu_t = \mathcal{T}_6(u, u, u, u, u), \tag{1.4.5}$$

where $\mathcal{T}_6$ is defined by the simple relation,

$$\langle \mathcal{T}_6(f_1, f_2, f_3, f_4, f_5), g\rangle_{L^2} = 6\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, g). \tag{1.4.6}$$

Theorem 1.4.1. *There holds the representation,*

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)(x) = \frac{12}{\pi}\int_{-\pi/4}^{\pi/4} e^{-itH}\left[(e^{itH}f_1)(e^{itH}f_2)(e^{itH}f_3)\overline{(e^{itH}f_4)(e^{itH}f_5)}\right](x)dt. \tag{1.4.7}$$

Proof. Using (1.4.6) we have,

$$\begin{aligned}
\langle \mathcal{T}_6(f_1, f_2, f_3, f_4, f_5), g\rangle_{L^2} &= 6\mathcal{E}(f_1, f_2, f_3, f_4, f_5, g) \\
&= \frac{12}{\pi}\int_0^{\pi/2} \left\langle (e^{itH}f_1)(e^{itH}f_2)(e^{itH}f_3)\overline{(e^{itH}f_4)(e^{itH}f_5)}, e^{itH}g)\right\rangle_{L^2(\mathbb{R})} dt \\
&= \frac{12}{\pi}\int_0^{\pi/2} \left\langle e^{-itH}\left[(e^{itH}f_1)(e^{itH}f_2)(e^{itH}f_3)\overline{(e^{itH}f_4)(e^{itH}f_5)}\right], g)\right\rangle_{L^2(\mathbb{R})} dt,
\end{aligned}$$

where we have used the fact that e^{itH} is an isometry of L^2 with inverse e^{-itH}. Upon commuting the space and time integrations, we get (1.4.7). □

This Theorem shows that the Hamiltonian flow corresponding to $\mathcal{H}_6$ is precisely the resonant equation 1.2.9 in the quintic case $k = 2$, up to a rescaling of time. By the approximation result, Theorem 1.2.3, solutions of (1.4.5) with initial data of size ϵ are close to solutions of $iu_t - \Delta u + |x|^2 = |u|^4 u$ in the space $\mathcal{H}^s$ for $s > 1/2$ and times $t \lesssim \epsilon^{-5}$.

1.4.1 · Representations of the Hamiltonian and the flow operator

We discussed in the introduction that a useful approach to the study of Hamiltonians such as $\mathcal{H}_6$ is to determine alternative representation formulas for $\mathcal{H}_6$, $\mathcal{E}_6$, and $\mathcal{T}_6$. Functionals such as $\mathcal{E}_6$ can have a large amount of structure that is concealed by a specific representations such as (1.4.2). This is will be illustrated clearly below.

First, we show that $\mathcal{E}_6$ is invariant under the Fourier transform. In fact, this invariance will be a simple consequence of Theorem 1.3.2 (which showed Fourier transform invariance for E_A) once we have established a specific representation of $\mathcal{E}_6$ in Theorem 1.4.7 below. However we wish to employ Fourier transform invariance before establishing that representation.

Lemma 1.4.2. *The functional $\mathcal{E}_6$ and operator $\mathcal{T}_6$ are invariant under the Fourier transform,*

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \mathcal{E}_6(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5, \widehat{f}_6), \tag{1.4.8}$$

$$\widehat{\mathcal{T}_6}(f_1, f_2, f_3, f_4, f_5) = \mathcal{T}_6(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5). \tag{1.4.9}$$

Proof. First let $f_k = \phi_{n_k}$ be Hermite functions. Then $\hat{\phi}_{n_k} = (i)^{n_k}\phi_{n_k}$, and so,

$$\mathcal{E}_6(\hat{\phi}_{n_1}, \hat{\phi}_{n_2}, \hat{\phi}_{n_3}, \hat{\phi}_{n_4}, \hat{\phi}_{n_5}, \hat{\phi}_{n_6}) = (i)^{n_1+n_2+n_3-n_4-n_5-n_6}\mathcal{E}_6(\phi_{n_1}, \phi_{n_2}, \phi_{n_3}, \phi_{n_4}, \phi_{n_5}, \phi_{n_6}). \tag{1.4.10}$$

On the other hand,

$$\begin{aligned}\mathcal{E}_6(\phi_{n_1}, \dots, \phi_{n_6}) = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} & e^{2i(n_1+n_2+n_3-n_4-n_5-n_6)}dt \\ & \times \int_{\mathbb{R}} \phi_{n_1}(x)\phi_{n_1}(x)\phi_{n_2}(x)\phi_{n_3}(x)\overline{\phi_{n_4}(x)\phi_{n_5}(x)\phi_{n_6}(x)}dx\end{aligned} \tag{1.4.11}$$

If $n_1 + n_2 + n_3 - n_4 - n_5 - n_6$ is a nonzero even integer, then the time integral in (1.4.11) is 0. If $n_1 + n_2 + n_3 - n_4 - n_5 - n_6$ is an odd integer, then by the Hermite function property $\phi_{n_k}(-x) = (-1)^{n_k}\phi_n(x)$, the integrand in the space integral in (1.4.11) is odd and hence the integral is 0. Therefore, using also (1.4.10), if $n_1 + n_2 + n_3 - n_4 - n_5 - n_6 \neq 0$, both $\mathcal{E}(\hat{\phi}_{n_1}, \dots, \hat{\phi}_{n_6})$ and $\mathcal{E}(\phi_{n_1}, \dots, \phi_{n_6})$ are 0 and in particular equal. Moreover, if $n_1 + n_2 + n_3 - n_4 - n_5 - n_6 = 0$, then by (1.4.10) $\mathcal{E}(\hat{\phi}_{n_1}, \dots, \hat{\phi}_{n_6}) = \mathcal{E}(\phi_{n_1}, \dots, \phi_{n_6})$.

Because the Hermite functions are a basis of L^2, the formula (1.4.8) holds for all functions f_k. The operator statement then follows from the same computation (1.3.12) as in the proof of Theorem 1.3.2. □

Theorem 1.4.3. *There holds the representations,*

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} (e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)\overline{(e^{it\Delta}f_4)(e^{it\Delta}f_5)(e^{it\Delta}f_6)}dxdt, \tag{1.4.12}$$

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)(x) = \frac{12}{\pi}\int_{-\pi/4}^{\pi/4} e^{-it\Delta}\left[(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)\overline{(e^{it\Delta}f_4)(e^{it\Delta}f_5)}\right](x)dt. \tag{1.4.13}$$

Proof. Recall that the lens transform (1.3.13) takes solutions u of the linear Schrödinger equation into solutions v of the linear Schrödinger equation with harmonic trapping. If we let $u_k(x,t) = (e^{itH}f_k)(x)$ and $v_k(x,t) = (e^{it\Delta}f_k)(x)$, the lens transform reads,

$$u_k(x,t) = \frac{1}{\cos(2t)^{1/2}}v_k\left(\frac{x}{\cos(2t)}, \frac{\tan(2t)}{2}\right)e^{ix^2\tan(2t)/2}.$$

We substitute these expressions into (1.4.2) and perform two changes of variable. In the time variable, we perform $s = \frac{1}{2}\tan(2t)$. This change of variables bijectively maps $(-\pi/4, \pi/4)$ to $(-\infty, \infty)$ and has determinant $\cos(2t)^{-2}$.

In the space variable we perform $y = x/\cos(t)$; this has determinant $|\cos(2t)|^{-1}$. Then,

$$\begin{aligned}\int_{-\pi/4}^{\pi/4}\int_{\mathbb{R}} &(u_1u_2u_3\overline{u_4u_5u_6})(x,t)dxdt \\ &= \int_0^{\pi/2}\frac{1}{|\cos(2t)|^3}\int_{\mathbb{R}}(v_1v_2v_3\overline{v_4v_5v_6})\left(\frac{x}{\cos(2t)},\frac{\tan(2t)}{2}\right)(x,t)dxdt \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}}(v_1v_2v_3\overline{v_4v_5v_6})(y,s)(x,t)dyds,\end{aligned}$$

which gives (1.4.12). The expression for $\mathcal{T}_6$ follows from the same argument as in the proof of Theorem 1.4.1. □

Theorem 1.4.4. *Let $\Omega_1(x) = y_1 + y_2 + y_3 - y_4 - y_5 - x$ and $\Omega_2(x) = y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - x^2$. Then there holds the representations,*

$$\mathcal{E}_6(f_1,f_2,f_3,f_4,f_5,f_6) = \frac{1}{\pi^2}\int_{\mathbb{R}^6} f_1(y_1)f_2(y_2)f_3(y_3)\overline{f_4(y_4)f_5(y_5)f_6(y_6)}\delta_{\Omega_1(y_6)}\delta_{\Omega_2(y_6)}dy, \tag{1.4.14}$$

$$\mathcal{T}_6(f_1,f_2,f_3,f_4,f_5)(x) = \frac{6}{\pi^2}\int_{\mathbb{R}^5} f_1(y_1)f_2(y_2)f_3(y_3)\overline{f_4(y_4)f_5(y_5)}\delta_{\Omega_1(x)}\delta_{\Omega_2(x)}dy. \tag{1.4.15}$$

Proof. We evaluate (1.4.12) using the fundamental solution formula for the linear Schrödinger equation, $(e^{it\Delta}f_k)(x) = (4\pi it)^{-1/2}\int_{\mathbb{R}} e^{i|x-y_k|^2/4t}f_k(y_k)dy_k$. We have,

$$\mathcal{E}_6(f_1,f_2,f_3,f_4,f_5,f_6) \tag{1.4.16}$$

$$\begin{aligned}&= \frac{2}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)\overline{(e^{it\Delta}f_4)(e^{it\Delta}f_5)(e^{it\Delta}f_6)}dxdt \\ &= \frac{1}{32\pi^4}\int_{\mathbb{R}}\frac{1}{t^3}\int_{\mathbb{R}}\int_{\mathbb{R}^6} e^{-ix\Omega_1(y_6)/2t}e^{+i\Omega_2(y_6)/4t}f_1(y_1)f_2(y_2)f_3(y_3)\overline{f}_4(y_4)\overline{f}_5(y_5)\overline{f}_6(y_6) \\ &\qquad dydxdt \\ &= \frac{1}{4\pi^4}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^6} e^{+ix\Omega_1(y_6)}e^{+is\Omega_2(y_6)}f_1(y_1)f_2(y_2)f_3(y_3)\overline{f}_4(y_4)\overline{f}_5(y_5)\overline{f}_6(y_6)dydxds \\ &= \frac{1}{2\pi^3}\int_{\mathbb{R}}\int_{\mathbb{R}^6} e^{+is\Omega_2(y_6)}f_1(y_1)f_2(y_2)f_3(y_3)\overline{f}_4(y_4)\overline{f}_5(y_5)\overline{f}_6(y_6)\delta_{\Omega_1(y_6)}dydx \qquad (1.4.17) \\ &= \frac{1}{\pi^2}\int_{\mathbb{R}^6} f_1(y_1)f_2(y_2)f_3(y_3)\overline{f}_4(y_4)\overline{f}_5(y_5)\overline{f}_6(y_6)\delta_{\Omega_1(y_6)}\delta_{\Omega_2(y_6)}dy,\end{aligned}$$

which is (1.4.14). Equation (1.4.15) follows immediately from definition (1.4.6) with the L^2 inner product integration in y_6. □

Theorem 1.4.5. *There holds the representations,*

$$\begin{aligned}\mathcal{E}_6(f_1,f_2,f_3,f_4,f_5,f_6) = \frac{1}{2\pi^2}\int_{\mathbb{R}^6} &f_1(\beta+\xi)f_2(\lambda\beta+\gamma)f_3(\lambda\gamma+\xi-\lambda\xi) \\ &\overline{f}_4(\lambda\beta+\xi)\overline{f}_5(\beta+\lambda\gamma+\xi-\lambda\xi)\overline{f}_6(\gamma)d\beta d\eta d\xi d\gamma.\end{aligned} \tag{1.4.18}$$

$$\begin{aligned}\mathcal{T}_6(f_1,f_2,f_3,f_4,f_5)(x) = \frac{3}{\pi^2}\int_{\mathbb{R}^6} &f_1(\beta+\xi)f_2(\lambda\beta+x)f_3(\lambda x+\xi-\lambda\xi) \\ &\overline{f}_4(\lambda\beta+\xi)\overline{f}_5(\beta+\lambda x+\xi-\lambda\xi)d\beta d\eta d\xi.\end{aligned} \tag{1.4.19}$$

Proof. We start with formula (1.4.17). Introduce new variables $\alpha, \beta, \gamma, \eta, \xi$ by $y_1 = \beta + \xi$, $y_2 = \eta + \gamma$, $y_3 = \alpha$, $y_4 = \eta + \xi$ and $y_5 = \alpha + \beta$. We calculate $y_6 = y_1 + y_2 + y_3 - y_4 - y_5 = \gamma$ and

$$\Omega_2(y_6) = y_1^2 + y_2^2 + y_3^2 - y_4^2 - y_5^2 - y_6^2 = 2\beta\xi + 2\gamma\eta - 2\eta\xi - 2\alpha\beta,$$

which gives the formula,

$$\begin{aligned}\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^3} \int_{\mathbb{R}^6} e^{2it[\beta\xi+\gamma\eta-\eta\xi-\alpha\beta]} f_1(\beta+\xi) f_2(\eta+\gamma) f_3(\alpha) \\ \overline{f}_4(\eta+\xi)\overline{f}_5(\alpha+\beta)\overline{f}_6(\gamma) d\alpha d\beta d\gamma d\eta d\xi dt\end{aligned}$$

Now change variables from η to λ through $\eta = \lambda\beta$. This gives $d\eta = |\beta| d\lambda$ and therefore,

$$\begin{aligned}\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^3} \int_{\mathbb{R}^6} |\beta| e^{2it\beta[\xi+\gamma\lambda-\xi\lambda-\alpha]} f_1(\beta+\xi) f_2(\lambda\beta+\gamma) f_3(\alpha) \\ \overline{f}_4(\lambda\beta+\xi)\overline{f}_5(\alpha+\beta)\overline{f}_6(\gamma) d\alpha d\beta d\gamma d\eta d\xi dt.\end{aligned}$$

Next we use the Fourier inversion formula $\int_{\mathbb{R}}\int_{\mathbb{R}} e^{iatx}\phi(x)dxdt = 2\pi|a|^{-1}\phi(0)$, with $a = 2\beta$ and $x(\alpha) = \xi + \gamma\lambda - \xi\lambda - \alpha$. This gives,

$$\begin{aligned}\mathcal{E}_A(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}^6} f_1(\beta+\xi) f_2(\lambda\beta+\gamma) f_3(\xi+\lambda\gamma-\lambda\xi) \\ \overline{f}_4(\lambda\beta+\xi)\overline{f}_5(\beta+\xi+\lambda\gamma-\lambda\xi)\overline{f}_6(\gamma) d\beta d\eta d\xi d\gamma,\end{aligned}$$

which is (1.4.18). The representation (1.4.19) follows from the definition of $\mathcal{T}_6$ in (1.4.6) with the L^2 inner product integration in γ. ☐

Theorem 1.4.6. *There holds the representations,*

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}} \frac{1}{\lambda^2-\lambda+1} E_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5, f_6) d\lambda, \tag{1.4.20}$$

$$\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}} \frac{1}{\lambda^2-\lambda+1} T_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5)(x) d\lambda. \tag{1.4.21}$$

where, for all λ, $A(\lambda)$ is an isometry and $A(\lambda)(1,1,1) = (1,1,1)$. (The matrix $A(\lambda)$ is given explicitly in (1.4.22).*)*

Proof. In formula (1.4.18), let y_1, y_2, y_3 be the arguments of f_1, f_2, f_3 respectively, and let x_1, x_2, x_3 be the arguments of $\overline{f}_4, \overline{f}_5, \overline{f}_6$ respectively. We have,

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \beta+\xi \\ \lambda\beta+\gamma \\ \lambda\gamma+\xi-\lambda\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ \lambda & 1 & 0 \\ 0 & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix} := B(\lambda) \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix},$$

and,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda\beta+\xi \\ \beta+\lambda\gamma+\xi-\lambda\xi \\ \gamma \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 1 \\ 1 & \lambda & 1-\lambda \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix} := C(\lambda) \begin{pmatrix} \beta \\ \gamma \\ \xi \end{pmatrix}.$$

In equation (1.4.18), perform the linear change of variables $x = C(\lambda)(\beta, \gamma, \xi)$. We find that $\det C(\lambda) = \lambda^2 - \lambda + 1 =$

$(\lambda - \frac{1}{2})^2 + \frac{3}{4} > 0$; in particular $C(\lambda)^{-1}$ is defined for all λ. Let $A(\lambda) = B(\lambda)C(\lambda)^{-1}$. Changing variables then establishes (1.4.20). The expression for $\mathcal{T}_6$ follows using the definition of T_A in Definition 1.3.1.

A calculation reveals that,

$$A(\lambda) = B(\lambda)C(\lambda)^{-1}$$

$$= \frac{1}{\lambda^2 - \lambda + 1} \begin{pmatrix} \lambda & 1-\lambda & \lambda^2 - \lambda \\ \lambda^2 - \lambda & \lambda & 1 - \lambda \\ 1 - \lambda & \lambda^2 - \lambda & \lambda \end{pmatrix} \tag{1.4.22}$$

$$= \frac{1}{\lambda^2 - \lambda + 1} \left[\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \lambda(\lambda - 1) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right]. \tag{1.4.23}$$

It remains to verify the two properties of $A(\lambda)$. These can, of course, be determined from the formula (1.4.22); however it is more insightful to see how they arise naturally from the combinatorical structure of the arguments to the functions in (1.4.18).

(i) By inspecting (1.4.18), we find that the squares of the arguments in f_1, f_2, f_3 sum to the squares of the arguments in f_4, f_5, f_6,

$$(\beta + \xi)^2 + (\lambda\beta + \gamma)^2 + (\lambda\gamma + \xi - \lambda\xi)^2 = (\lambda\beta)^2 + (\beta + \lambda\gamma + \xi - \lambda\xi)^2 + (\gamma)^2. \tag{1.4.24}$$

This gives, for all $x \in \mathbb{R}^3$, that $|B(\lambda)x|^2 = \sum_{k=1}^3 |\langle B(\lambda), e_k \rangle|^2 = \sum_{k=1}^3 |\langle C(\lambda), e_k \rangle|^2 = |C(\lambda)x|^2$. Setting $x = C(\lambda)^{-1}y$ gives $|A(\lambda)y|^2 = |y|^2$ for all $y \in \mathbb{R}^3$, and hence $A(\lambda)$ is an isometry.

(ii) Again in (1.4.18), we see that the arguments in f_1, f_2, f_3 sum to the arguments in f_4, f_5, f_6,

$$(\beta + \xi) + (\lambda\beta + \gamma) + (\lambda\gamma + \xi - \lambda\xi) = (\lambda\beta) + (\beta + \lambda\gamma + \xi - \lambda\xi) + (\gamma) \tag{1.4.25}$$

Setting $e = (1,1,1)$, this means that for all x, $\langle B(\lambda)x, e \rangle = \langle C(\lambda)x, e \rangle$. Set $y = C(\lambda)x$ to give $\langle A(\lambda)y, e \rangle = \langle y, e \rangle$. Because A is an isometry, $A^* = A^{-1}$, and so $\langle y, A^{-1}e \rangle = \langle y, e \rangle$ for all y, and hence $Ae = e$.

We note that the expressions (1.4.24) and (1.4.25) arise naturally from the δ arguments in (1.4.14). The properties of $A(\lambda)$ in (i) and (ii) should thus be considered generic for continuous resonant type equations. □

Theorem 1.4.7. *We have the representations,*

$$\mathcal{E}(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6) d\theta, \tag{1.4.26}$$

$$\mathcal{T}(f_1, f_2, f_3, f_4, f_5)(x) = \frac{\sqrt{3}}{\pi^2} \int_0^{2\pi} T_{R(\theta)}(f_1, f_2, f_3, f_4, f_5)(x) d\theta, \tag{1.4.27}$$

where $R(\theta)$ is the rotation of θ radians about the axis $(1,1,1)$.

Proof. Because the matrix $A(\lambda)$ is an isometry, $\det A(\lambda) = +1$, and $A(\lambda)(1,1,1) = (1,1,1)$, the matrix must, in fact, be a rotation about the axis $(1,1,1)$. For any rotation A of $\mathbb{R}^3$, the angle of rotation θ satisfies, $2\cos(\theta) + 1 =$

Trace(A). In the present case, this means,

$$\cos(\theta) = \phi(\lambda) := \frac{1}{2}\left(\mathrm{Trace}(A(\lambda)) - 1\right) = \frac{1}{2}\left(\frac{3\lambda}{\lambda^2 - \lambda + 1} - 1\right). \tag{1.4.28}$$

The formula (1.4.26) follows from performing the change of variables $\lambda \mapsto \theta$, which we do rather carefully.

From analyzing (1.4.28), we determine that ϕ has the following properties. It satisfies $\phi(-1) = -1$, $\phi(1) = 1$; ϕ is increasing on $[-1, 1]$; ϕ is decreasing on $(-\infty, -1] \cup [1, \infty)$; and $\lim_{\lambda \to -\infty} \phi(\lambda) = \lim_{\lambda \to +\infty} \phi(\lambda) = 0$. By setting $\lambda = 0$ in (1.4.23), we determine that $\sin(\theta) = -1\sqrt{3} < 0$, and hence $\theta = -4\pi/3 \in [\pi, 2\pi]$. On the other hand, as $\lambda \to \pm\infty$, $\sin(\theta) \to +1/\sqrt{3} > 0$. From these considerations and continuity, we infer that that under $\lambda \mapsto \theta$, $[-1, 1]$ is bijectively mapped to $[\pi, 2\pi]$, while $(-\infty, -1) \cup (1, \infty)$ is bijectively mapped to $(0, \pi)$. In all, $\mathbb{R}$ is bijectively mapped to $(0, 2\pi]$.

To perform the change of variables, we need to find the determinent which is given by,

$$\frac{d\theta}{d\lambda} = \left|\frac{d}{d\lambda}\arccos\left(\frac{1}{2}\left[\frac{3\lambda}{\lambda^2 - \lambda + 1} - 1\right]\right)\right|.$$

To simplify the computation, we first find that if $a = \sqrt{3}(1 - \lambda)/(1 + \lambda)$, then

$$\begin{aligned}\arccos\left(\frac{1}{2}\left[\frac{3\lambda}{\lambda^2 - \lambda + 1} - 1\right]\right) &= \arccos\left(\frac{1 - a^2}{1 + a^2}\right) = \arctan\left(\frac{2a}{1 - a^2}\right) \\ &= 2\arctan(a) = 2\arctan\left(\frac{\sqrt{3}(1 - \lambda)}{1 + \lambda}\right);\end{aligned}$$

we then calculate $|d\theta/d\lambda| = \sqrt{3}(\lambda^2 - \lambda + 1)^{-1}$. Formula (1.4.26) then follows. □

We have derived, in total, six different representations for each of $\mathcal{E}_6$, $\mathcal{H}_6$ and $\mathcal{T}_6$. Before moving on, we give a sketch of the relevance of each of these representations. The first expression (1.4.2), in terms of e^{itH}, shows that the Hamiltonian system $\mathcal{H}_6$ is the resonant system for $k = 2$ discussed in Section 1.2. It also illustrates clearly the permutation symmetries (1.4.3) and (1.4.4) which are far from obvious from the last three representations in Theorems 1.4.5, 1.4.6 and 1.4.7. The representations (1.4.12) (in terms of $e^{it\Delta}$) and (1.4.14) will not be used extensively in this chapter, but they very importantly show that $\mathcal{H}_6$ is the continuous resonant system for $d = 1$ and $p = 2$ discussed in [12]. In the remainder of this section we will mostly use the last three representations. The expression (1.4.18) is useful for proving refined multilinear estimates: if we have information about the supports of the functions f_k then we can restrict the range of integration and prove smallness, in tandem with (1.4.20). The last representation (1.4.26) makes determining certain properties of $\mathcal{E}_6$, like L^2 boundedness and symmetry structures, almost trivial because we know so much about $E_{R(\theta)}$ from Section 1.3. The following two subsections are largely about importing such information about E_A from Section 1.3 to $\mathcal{E}_6$.

1.4.2 · Symmetries of the Hamiltonian and conserved quantities of the flow

Theorem 1.4.8. *The functional $\mathcal{E}_4(f_1, f_2, f_3, f_4, f_5, f_6)$ is invariant under the following actions (for all λ).*

(i) Fourier transform, $f_k \mapsto \widehat{f}_k$.

(ii) Modulation, $f_k \mapsto e^{i\lambda} f_k$.

(iii) L^2 scaling, $f_k(x) \mapsto \lambda^{1/2} f_k(\lambda x)$.

(iv) Linear modulation, $f_k \mapsto e^{i\lambda} f_k$.

(v) Translation, $f_k \mapsto f_k(\cdot + \lambda)$.

(vi) Quadratic modulation, $f_k \mapsto e^{i\lambda x^2} f_k$.

(vii) Schrödinger group, $f_k \mapsto e^{i\lambda\Delta} f_k$.

(viii) Schrödinger with harmonic trapping group, $f_k \mapsto e^{i\lambda H} f_k$.

Proof. Each of the symmetries commutes with the integration over θ in (1.4.26). The result then follows immediately from Theorems 1.3.2 and 1.3.3. □

Corollary 1.4.9. *We have the following commuter equalities,*

$$\begin{aligned} e^{i\lambda Q}\mathcal{T}(f_1, f_2, f_3, f_4, f_5) &= \mathcal{T}(e^{i\lambda Q} f_1, e^{i\lambda Q} f_2, e^{i\lambda Q} f_3, e^{i\lambda Q} f_4, e^{i\lambda Q} f_5) \quad (1.4.29) \\ Q\mathcal{T}(f_1, f_2, f_3, f_4, f_5) &= \mathcal{T}(Qf_1, f_2, f_3, f_4, f_5) + \mathcal{T}(f_1, Qf_2, f_3, f_4, f_5) \\ &\quad + \mathcal{T}(f_1, f_2, Qf_3, f_4, f_5) \\ &\quad - \mathcal{T}(f_1, f_2, f_3, Qf_4, f_5) - \mathcal{T}(f_1, f_2, f_3, f_4, Qf_5) \quad (1.4.30) \end{aligned}$$

where Q *are the operators:* $Q = 1$, $Q = x$, $Q = id/dx$, $Q = x^2$, $Q = \Delta$, $Q = H$.

Proof. These follow immediately from the representation (1.4.27) and Theorem 1.3.4. □

By Theorem, 1.3.5, the Hamiltonian flow associated to $\mathcal{H}_6$ has conserved quantities for symmetries (ii) through (viii). These symmetries and conserved quantities and summarized in the following table.

Symmetry of $\mathcal{H}_6$	Conserved quantity	Operator commuting with $\mathcal{T}_6$
$f \mapsto e^{i\lambda} f$	$\int_{\mathbb{R}} \lvert f(x)\rvert^2 dx$	1
$f \mapsto f_\lambda$	$\int_{\mathbb{R}} [ixf'(x) + f(x)]\,\overline{f}(x)dx$	
$f \mapsto e^{i\lambda x} f$	$\int_{\mathbb{R}} x\lvert f(x)\rvert^2 dx$	x
$f \mapsto f(\cdot + \lambda)$	$\mathrm{Re} \int_{\mathbb{R}} f'(x)\overline{f}(x)dx$	id/dx
$f \mapsto e^{i\lambda \lvert x\rvert^2} f$	$\int_{\mathbb{R}} \lvert xf(x)\rvert^2 dx$	x^2
$f \mapsto e^{i\lambda\Delta} f$	$\int_{\mathbb{R}} \lvert f'(x)\rvert^2 dx$	Δ
$f \mapsto e^{i\lambda\mathcal{H}} f$	$\int_{\mathbb{R}} \lvert xf(x)\rvert^2 + \lvert f'(x)\rvert^2 dx$	H

1.4.3 · Boundedness of the functional and wellposedness of Hamilton's equation

Theorem 1.4.10. *There holds the following sharp bound,*

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{\pi\sqrt{3}} \prod_{k=1}^{6} \|f_k\|_{L^2}; \quad (1.4.31)$$

which means in particular $0 \leq \mathcal{H}_6(f) \leq 1/(\pi\sqrt{3})\|f\|_{L^2}^6$. *Equality holds in* (1.4.31) *if and only if each* f_k *is the same Gaussian* $ce^{-\alpha x^2+\beta x}$ *for some* $c, \alpha, \beta \in \mathbb{C}$ *and* $Re\,\alpha > 0$.

Proof. Using representation (1.4.26) and Theorem 1.3.6 we have,

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{2\sqrt{3}\pi^2}\int_0^{2\pi} |E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6)| d\theta \leq \frac{1}{\pi\sqrt{3}}\prod_{k=1}^{6} \|f_k\|_{L^2},$$

which is the inequality (1.4.31).

For equality to hold, we must have equality in,

$$|E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6)| = \prod_{k=1}^{6} \|f_k\|_{L^2}, \tag{1.4.32}$$

for almost every $\theta \in [0, 2\pi]$. We know from Theorem 1.3.12 that if the functions f_k are the same Gaussian $e^{-\alpha x^2+\beta x}$ for $\alpha, \beta \in \mathbb{C}$ and $\operatorname{Re}\alpha > 0$ then equality does hold. (This uses the fact that $R(\theta)$ is an isometry for all θ and that $R(\theta)(1, 1, 1) = (1, 1, 1)$.)

On the other hand, because $R(\theta)$ is not a signed permutation matrix almost everywhere, Theorem 1.3.12 says that any functions f_k satisfying (1.4.32) must be Gaussians. Write $f_k(x) = e^{-\alpha_k x^2+\beta_k x}$ for $\alpha_k, \beta_k \in \mathbb{C}$ with $\operatorname{Re}\alpha_k > 0$. The Cauchy–Schwarz equality condition (1.3.20) in this case reads,

$$f_1((R(\theta)x)_1) f_2((R(\theta)x)_2) f_3((R(\theta)x)_3) = f_4(x_1) f_5(x_2) f_6(x_3), \tag{1.4.33}$$

Substituting the expressions for f_k gives, for all $\theta \in [0, 2\pi]$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\sum_{k=1}^{3} -\alpha_k (R(\theta)x)_k^2 + \beta_k (R(\theta)x)_k = \sum_{k=1}^{3} -\alpha_{3+k} x_k^2 + \beta_{3+k} x_k.$$

Setting variously $x = (1, 0, 0)$, $x = (0, 1, 0)$ and $x = (0, 0, 1)$, along with $\theta = 0$, $\theta = 2\pi/3$ and $\theta = 4\pi/3$ gives that $\alpha_k = \alpha_1$ and $\beta_k = \beta_1$ for all k. □

Using the expression $\mathcal{H}_6(f) = (2/\pi)\|e^{it\Delta} f\|_{L^6}^6$ from Theorem 1.4.3, one obtains the sharp inequality,

$$\|e^{it\Delta} f\|_{L^6}^6 \leq \frac{1}{2\sqrt{3}}\|f\|_{L^2}^6, \tag{1.4.34}$$

which is the Strichartz inequality with best constant in dimension one, as previously proved in [20]

Theorem 1.4.11. *We have the operator bound $\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_X \leq C_X \prod_{k=1}^{5} \|f_k\|_X$ for the following spaces.*

(i) $X = L^2$ *with* $C_X = 2\sqrt{3}/\pi$,

(ii) $X = L^{2,\sigma}$, *for any* $\sigma \geq 0$.

(iii) $X = H^\sigma$, *for any* $\sigma \geq 0$.

(iv) $X = L^{\infty,s}$, *for any* $s > 1/2$.

(v) $X = L^{p,s}$, *for any* $p \geq 2$ *and* $s > 1/2 - 1/p$.

Proof. The bounds (i), (ii), and (iii) follow immediately from Theorem 1.3.7 using the representation of $\mathcal{T}_6$ in (1.4.27).

The bound (iv) is proved in Theorem 1.4.18 below.

The bound (v) comes from interpolating between the bounds in (iv) and (ii). □

Theorem 1.4.12. *Consider the Cauchy problem,*

$$\begin{aligned} iu_t &= \mathcal{T}_6(u,u,u,u,u), \\ u(t=0) &= u_0, \end{aligned} \tag{1.4.35}$$

which is Hamilton's equation corresponding to $\mathcal{H}_6$ and the resonant equation (1.1.4) *in the quintic $k=2$ case.*

(i) The Cauchy problem (1.4.35) *is locally well-posed in X for any of the spaces X in Theorem 1.4.11.*

(ii) The Cauchy problem (1.4.35) *is globally well-posed in L^2.*

Proof. (i) This follows from Theorem 1.3.8, using the bounds on $\mathcal{T}_6$ established in the previous theorem.

(ii) We know from Theorem 1.3.8 that the local time of existence of a solution to (1.4.35) in L^2 depends only on $\|u_0\|_{L^2}$. Because $\|u\|_{L^2}$ is conserved by the flow (1.4.35), by the usual argument the L^2 solution is global. □

1.4.4 · Analysis of the stationary waves

Stationary wave solutions are solutions of the form $e^{i\omega t}\psi(x)$ for some $\omega \in \mathbb{R}$ and a function ψ. By substitution into (1.4.5), we find that ψ must satisfy,

$$-\omega\psi(x) = \mathcal{T}_6(\psi,\psi,\psi,\psi,\psi)(x) = \frac{\sqrt{3}}{\pi^2}\int_0^{2\pi} T_{R(\theta)}(\psi,\psi,\psi,\psi,\psi)(x)d\theta. \tag{1.4.36}$$

In Theorem 1.3.10, we found that if ϕ_n is a Hermite function then $T_A(\phi_n,\phi_n,\phi_n,\phi_n,\phi_n)(x) = C\phi_n(x)$, which means from (1.4.36) that $\mathcal{T}_6(\phi_n,\dots,\phi_n)(x) = C\phi_n(x)$ for some constant C. The function ϕ_n is thus a stationary wave. By letting the symmetries of $\mathcal{T}_6$ act on ϕ_n, we find that each of the functions,

$$ae^{ibx+icx^2}\phi_n(dx+e), \tag{1.4.37}$$

for $a \in \mathbb{C}$ and $b,c,d,e \in \mathbb{R}$ is a stationary wave solution of (1.4.5).

1.4.4.1 · Regularity of stationary waves: introduction

All of the stationary waves (1.4.37) are analytic and decay in space like $e^{-\alpha x^2}$ for some $\alpha \in \mathbb{R}$. The remainder of this section is devoted to a proof *any* function $\psi \in L^2$ satisfying (1.4.36) is automatically analytic and exponentially decaying in space like $e^{-\alpha x^2}$. Our proof follows closely the proof of the analogous result for the two-dimensional continuous resonant equation in [25], which in turn is based on work in [33]; there are also similar results in [18, 28].

Our proof here has two main ingredients. Roughly speaking, once a multilinear functional can supply these ingredients, the associated Hamiltonian system will satisfy a result like Theorem 1.4.15 below. The first ingredient is an ability to transfer exponential weight from one input of the functional to the other inputs, as in Lemma

1.3.13 for the functionals E_A. The second ingredient is a refined multilinear estimate, which we discuss in the next subsection.

For the weight transfer property, recall that in Lemma 1.3.13, we defined $G_{\mu,\epsilon}(x) = \exp(\mu x^2/(1+\epsilon x^2))$ and established that if A is an isometry and functions f_k are non-negative, then,

$$E_A(f_1, \dots, f_{2n-1}, f_{2n} G_{\mu,\epsilon}) \le E_A(f_1 G_{\mu,\epsilon}, \dots, f_{2n-1} G_{\mu,\epsilon}, f_{2n}).$$

This property is immediately inherited by $\mathcal{E}_6$: if functions f_k are non-negative, then,

$$\begin{aligned}
&\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6 G_{\mu,\epsilon}) \\
&\quad = \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6 G_{\mu,\epsilon}) d\theta \\
&\quad \le \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_{R(\theta)}(f_1 G_{\mu,\epsilon}, f_2 G_{\mu,\epsilon}, f_3 G_{\mu,\epsilon}, f_4 G_{\mu,\epsilon}, f_5 G_{\mu,\epsilon}, f_6) d\theta \\
&\quad = \mathcal{E}_6(f_1 G_{\mu,\epsilon}, f_2 G_{\mu,\epsilon}, f_3 G_{\mu,\epsilon}, f_4 G_{\mu,\epsilon}, f_5 G_{\mu,\epsilon}, f_6).
\end{aligned} \tag{1.4.38}$$

The inequality (1.4.38) is the first ingredient in the proof.

1.4.4.2 · Refined multilinear Strichartz estimates

The second ingredient we need is a so-called refined multilinear Strichartz estimate. Such estimates are treated in a number of works [7, 9, 42]. Lemma 111 in [9] is prototypical of the type of estimate we require here: it states that if functions $f_1, f_2 \in L^2(\mathbb{R}^2 \to \mathbb{C})$ satisfy $\operatorname{supp} \widehat{f}_1 \subset B(0,N)$ and $\operatorname{supp} \widehat{f}_2 \subset B(0,M)^C$, with $N \ll M$, then,

$$\|(e^{it\Delta} f_1)(e^{it\Delta} f_2)\|_{L^4(\mathbb{R}^2 \times \mathbb{R})} \lesssim \left(\frac{N}{M}\right)^{1/2} \|f_1\|_{L^2(\mathbb{R}^2)} \|f_2\|_{L^2(\mathbb{R}^2)}.$$

The right hand side is decaying for large M and small N. In our case, under similar support assumptions on functions $\widehat{f}_i$ and $\widehat{f}_j$, we would like to have analogous control on,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3) \overline{(e^{it\Delta} f_4)(e^{it\Delta} f_5)(e^{it\Delta} f_6)} dx dt;$$

namely, we would like an L^2 bound that is decaying as the supports of $\widehat{f}_i$ and $\widehat{f}_j$ become increasingly disjoint. Using the representations (1.4.18) and (1.4.20) we are in fact able to determine the required refined mulitlinear estimate in an elementary way.

Because we know that $\mathcal{E}_6$ is invariant under the Fourier transform, it is equivalent to state the support assumptions in terms of f_i and f_j and not their Fourier transforms.

Proposition 1.4.13. *Suppose that the support of f_2 is in $B(0,R)^C$ and the supports of f_3, f_5 and f_6 are in $B(0,r)$, with $R > 4r$. Then,*

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \le \frac{1}{R} \prod_{k=1}^{6} \|f_k\|_{L^2}.$$

Proof. We use the representation of $\mathcal{E}_6$ given in (1.4.18),

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\pi^2} \int_{\mathbb{R}^6} f_1(\beta+\xi) f_2(\lambda\beta+\gamma) f_3(\lambda\gamma+\xi-\lambda\xi) \\ \overline{f}_4(\lambda\beta+\xi)\overline{f}_5(\beta+\lambda\gamma+\xi-\lambda\xi)\overline{f}_6(\gamma)\,d\beta d\eta d\xi d\gamma,$$

to identify a large set in λ on which the integrand is 0. We will then use the representation (1.4.20) to obtain L^2 bounds, recalling that the integrand as a function of λ is the same in both representations.

Under the assumptions of the proposition, the integrand is non-zero only when,

$$|\beta| \leq |\beta+\lambda\gamma+\xi-\lambda\xi| + |\lambda\gamma+\xi-\lambda\xi| \leq 2r,$$

and only when,

$$|\lambda\beta| \geq |\lambda\beta+\gamma| - |\gamma| \geq R - 2r \geq \frac{R}{2}.$$

It follows that the integrand is non-zero only when $|\lambda| > R/4$.

Then, using the representation (1.4.20),

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{2\pi^2} \int_{|\lambda|>R/4} \frac{1}{\lambda^2-\lambda+1} |E_{A(\lambda)}(f_1, f_2, f_3, f_4, f_5, f_6)| d\lambda \\ \leq \frac{1}{\pi^2}\left(\int_{|\lambda|>R/4} \frac{1}{\lambda^2-\lambda+1} d\lambda\right) \prod_{k=1}^{6} \|f_k\|_{L^2} \leq \frac{1}{R}\prod_{k=1}^{6} \|f_k\|_{L^2},$$

which is what we wanted to prove. □

Proposition 1.4.14. *Suppose that for some i and some j, the support of f_i is in $B(0,R)^C$ and the support of f_j is in $B(0,r)$, with $R > 4r$. Then,*

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{1}{R^{1/6}} \prod_{k=1}^{6} \|f_k\|_{L^2}. \tag{1.4.39}$$

Proof. We assume, by rescaling, that $\|f_k\|_{L^2} = 1$ for all k. We have the crude bound $\mathcal{H}_6(f) = \|e^{it\Delta} f_k\|_{L^6}^6 \leq 1$, from (1.4.34). Then,

$$|\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6)| \leq \frac{2}{\pi}\int_{\mathbb{R}^2} |e^{it\Delta} f_1| \cdot |e^{it\Delta} f_2| \cdot |e^{it\Delta} f_3| \cdot |e^{it\Delta} f_4| \cdot |e^{it\Delta} f_5| \cdot |e^{it\Delta} f_6| dx dt, \\ = \frac{2}{\pi}\left\|(e^{it\Delta} f_1)(e^{it\Delta} f_2)(e^{it\Delta} f_3)(e^{it\Delta} f_4)(e^{it\Delta} f_5)(e^{it\Delta} f_6)\right\|_{L^1} \\ \leq \frac{2}{\pi}\|(e^{it\Delta} f_i)(e^{it\Delta} f_j)\|_{L^3} = \frac{2}{\pi}\|(e^{it\Delta} f_i)^3(e^{it\Delta} f_j)^3\|_{L^1}^{1/3} \\ \leq \frac{2}{\pi}\|(e^{it\Delta} f_i)(e^{it\Delta} f_j)^2\|_{L^2}^{1/3}\|(e^{it\Delta} f_i)^2(e^{it\Delta} f_j)\|_{L^2}^{1/3} \\ \leq \left(\frac{2}{\pi}\right)^{2/3} \mathcal{E}_6(f_j, f_i, f_j, f_i, f_j, f_j)^{1/6} \leq \frac{1}{R^{1/6}},$$

which is (1.4.39). □

Using the weight transfer property (1.4.38) and the refined multilinear Strichartz estimate (1.4.39), we prove that stationary waves are necessarily analytic. We begin with an integrability result.

Theorem 1.4.15. *Suppose $\phi \in L^2$ satisfies,*

$$|\omega||\phi(x)| \le \mathcal{T}(|\phi|, |\phi|, |\phi|, |\phi|, |\phi|)(x). \tag{1.4.40}$$

Then there exists $\alpha > 0$ such that $x \mapsto \phi(x)e^{\alpha x^2} \in L^2$.

Proof of Theorem 1.4.15. For the proof, we will find μ so that we have the bound $\|\phi G_{\mu,\epsilon}\|_{L^2} \lesssim 1$ independently of ϵ. Taking the limit $\epsilon \to 0$ will the yield the result.

We can clearly assume that $\phi(x) \ge 0$, and will do so throughout. For any $M > 0$ define,

$$\phi_<(x) = \phi(x)\chi_{|x|\le M}(x), \qquad \phi_\sim(x) = \phi(x)\chi_{M<|x|\le M^2}(x), \qquad \phi_>(x) = \phi(x)\chi_{M^2<|x|}(x),$$

We have the decomposition $\phi = \phi_> + \phi_\sim + \phi_>$, and the supports are all disjoint, which gives,

$$\|\phi G_{\mu,\epsilon}\|_{L^2}^2 = \|\phi_< G_{\mu,\epsilon}\|_{L^2}^2 + \|\phi_\sim G_{\mu,\epsilon}\|_{L^2}^2 + \|\phi_> G_{\mu,\epsilon}\|_{L^2}^2.$$

The first two terms are trivial to deal with. If $|x| \le M^2$, we have,

$$G_{\mu,\epsilon}(x) \le \exp(\mu|x|^2) \le \exp(\mu M^4),$$

so setting $\mu \le M^{-4}$ gives $\|\phi_< G_{\mu,\epsilon}\|_{L^2} \le \|\phi_< e^1\|_{L^2} \le e\|\phi\|_{L^2}$, uniformly in ϵ. The same bound holds for $\phi_\sim$. It remains then to bound $\|\phi_> G_{\mu,\epsilon}\|_{L^2}$ uniformly in ϵ.

Starting with equation (1.4.40), we multiply both sides by $\phi_>(x)G_{\mu,\epsilon}(x)^2$,

$$\omega\phi(x)G_{\mu,\epsilon}(x)^2 \le \mathcal{T}(\phi, \dots, \phi)(x)\phi(x)G_{\mu,\epsilon}(x)^2.$$

Now integrating over $\mathbb{R}$, using the relationship between $\mathcal{E}_6$ and $\mathcal{T}_6$ in (1.4.6), and passing the exponential weight using (1.4.38), we determine the bound,

$$\omega\|\phi G_{\mu,\epsilon}\|_{L^2}^2 \le 6\mathcal{E}_6(\phi, \phi, \phi, \phi, \phi, \phi_> G_{\mu,\epsilon}{}^2) \lesssim \mathcal{E}_6(\phi e^{G_{\mu,\epsilon}}, \dots, \phi e^{G_{\mu,\epsilon}}, \phi_> e^{G_{\mu,\epsilon}}).$$

For convenience, let $\psi = \phi G_{\mu,\epsilon}$. The bound then reads,

$$\omega\|\psi\|_{L^2}^2 \lesssim \mathcal{E}_6(\psi, \psi, \psi, \psi, \psi, \psi_>). \tag{1.4.41}$$

Now write each $\psi = \psi_< + \psi_\sim + \psi_>$ and expand the multilinear functional. We will get many terms, which we bound in one of two ways.

- If there are three or more $\psi_>$ terms, bound by $\|\psi_>\|_{L^2}^k$ (where $k \ge 3$ is the number of $\psi_>$ terms appearing) using the standard L^2 bound (1.4.31). In this case the other terms are $\psi_<$ or $\psi_\sim$, which we know are uniformly bounded.
- If there are one or two $\psi_>$ terms, then there is either a $\psi_<$ term or a $\psi_\sim$ term. We may assume $M > 4$. Then in the former case we can use the refined multilinear estimate (1.4.39) (with $r = M$ and $R = M^2$)

and bound by $(1/M^{1/3})\|\psi_>\|_{L^2}^k$ (where $k = 1$ or $k = 2$). If there are no $\psi_<$ terms, we bound by $\|\psi_\sim\|_{L^2}\|\psi_>\|_{L^2}^k \lesssim \|\phi_\sim\|_{L^2}\|\psi_>\|_{L^2}^k$.

Using these, we find,

$$\begin{aligned}\omega\|\psi_>\|_{L^2}^2 &\leq 6\mathcal{E}_6(\psi,\psi,\psi,\psi,\psi,\psi_>)\\ &\leq C\left(\sum_{k=3}^{m}\|\psi_>\|^k + \left(\frac{1}{M^{1/3}} + \|\phi_\sim\|_{L^2}\right)(\|\psi_>\|_{L^2}^2 + \|\psi_>\|_{L^2})\right),\end{aligned} \tag{1.4.42}$$

for some constant C independent of ψ and ϵ. Set $\delta(M) = M^{-1/3} + \|\phi_\sim\|_{L^2}$ and,

$$x(\epsilon, M) = \|\psi_>\|_{L^2} = \|\phi\chi_{|x|<M^2}G_{\mu,\epsilon}\|_{L^2}.$$

Note that $\delta(M) \to 0$ as $M \to \infty$. Choose M sufficiently large so that $C\delta(M) \leq \omega/2$. This gives,

$$\frac{\omega}{2C}x(\epsilon, M)^2 \leq \sum_{k=3}^{m} x(\epsilon, M)^k + \delta(M)x(\epsilon, M).$$

Dividing through by $x(\epsilon, M) > 0$ and rearranging terms gives,

$$0 \leq p_{\delta(M)}(x(\epsilon, M)), \quad \text{where} \quad p_\delta(x) := \sum_{k=2}^{m-1} x^k - \frac{\omega}{2C}x + \delta. \tag{1.4.43}$$

Observe that $p_0(0) = 0$, $p_0'(0) = -\omega/2C < 0$ and $p_0(x) \to \infty$ as $x \to \infty$. This shows that p_0 has another 0 in $(0, \infty)$; call the smallest such zero x_0. The zeroes of a polynomial are continuous functions of the coefficients. Hence if we choose M sufficiently large we can assume that $p_{\delta(M)}$ has one zero in $(-\infty, x_0/3)$ (coming from $p_0(0) = 0$) and one zero $(2x_0/3, \infty)$ (coming from $p_0(x_0) = 0$) and that $p_{\delta(M)}(x) < 0$ in $(x_0/3, 2x_0/3)$. This shows that for all M sufficiently large,

$$Z_{\delta(M)} = p_{\delta(M)}^{-1}([0,\infty)) \subset (-\infty, x_0/3) \cup (2x_0/3, \infty).$$

Now we know from the inequality (1.4.43) that $x(\epsilon, M) \in Z_{\delta(M)}$ for all ϵ. If we set $\epsilon = 1$ we get,

$$x(1, M) = \|\psi_>\|_{L^2} = \|\phi_> e^{\mu x^2/(1+x^2)}\|_{L^2} \leq \|\phi_>\|_{L^2}e^{\mu}.$$

Recall that we set $\mu = M^{-4}$, so that $\mu \lesssim 1$ and so $x(1, M) \lesssim \|\phi_>\|_{L^2}$. As $M \to \infty$, $\|\phi_>\|_{L^2} = \|\phi\chi_{M^2<|x|}\|_{L^2} \to 0$, and hence if we take M sufficiently large we will have $x(1, M) \leq x_0/3$. But now because $x(\epsilon, M)$ depends continuously on ϵ, and $x(\epsilon, M) \in Z_{\delta(M)}$ for all ϵ, we have,

$$x(\epsilon, M) = \|\psi_>\|_{L^2} \leq x_0/3,$$

for all ϵ. Taking $\epsilon \to 0$ yields $x(0, M) = \|\phi_> e^{\mu x^2}\| \leq x_0/3 < \infty$, which is what we wanted to prove. □

Corollary 1.4.16. *Suppose that $\phi \in L^2$ is a stationary wave solution of the Hamiltonian flow associated to $\mathcal{H}_6$; that is, ϕ satisfies,*

$$\omega\phi(x) = \mathcal{T}_6(\phi,\phi,\phi,\phi,\phi)(x), \tag{1.4.44}$$

for some ω. Then there exists $\alpha > 0$ and $\beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\widehat{\phi} e^{\beta x^2} \in L^\infty$. As a result, ϕ can be extended to an entire function on the complex plane.

Proof. The condition (1.4.44) implies the condition (1.4.40) in the previous theorem, and hence there exists $\alpha > 0$ such that $\phi e^{2\alpha x^2} \in L^2$. Because $\mathcal{T}_6$ commutes with the Fourier transform, condition (1.4.44) also holds with ϕ replaced by $\hat{\phi}$. Then, again by the previous theorem, there exists $\beta > 0$ such that $\phi e^{2\beta x^2} \in L^2$.

To turn these L^2 bounds into L^∞ bounds, we first assume ϕ is Schwartz and compute,

$$\begin{aligned}\phi(x)^2 e^{2\alpha x^2} &= e^{2\alpha x^2} \int_x^\infty \frac{d}{dt}\phi(t)^2 dt = e^{2\alpha x^2} \int_x^\infty 2\phi(t)\phi'(t)dt \leq 2\int_x^\infty e^{2\alpha t^2}\phi(t)\phi'(t)dt \\ &\leq 2\|e^{2\alpha t^2}\phi(t)\|_{L^2}\|\phi'\|_{L^2} = 2\|e^{2\alpha t^2}\phi(t)\|_{L^2}\|\xi\hat{\phi}\|_{L^2} \\ &\leq \beta^{-1/2}\|e^{2\alpha t^2}\phi(t)\|_{L^2}\|e^{2\beta\xi^2}\hat{\phi}\|_{L^2},\end{aligned}$$

which gives $\phi(x)e^{\alpha x^2} \in L^\infty$. Because Schwartz functions are dense in L^2, this holds for arbitrary $\phi \in L^2$. The L^∞ bound for $\hat{\phi}$ follows similarly.

Finally, using the L^∞ bound $|\hat{\phi}(\xi)| \lesssim e^{-\beta\xi^2}$, and the inverse Fourier transform formula,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iz\xi}\hat{\phi}(\xi)d\xi.$$

we can extend ϕ to an entire function on the complex plane. □

1.4.5 · Smoothing and further boundedness properties

In this section we prove two further boundedness results for the operator $\mathcal{T}_6$. Unlike our previous boundedness results, which were simply inherited from the analogous results for T_A, the present results rely on additional structure in $\mathcal{T}_6$.

We first strengthen items (ii) and (iii) in Theorem 1.4.11. Item (ii) says that $\mathcal{T}_6$ is bounded from $(L^{2,\sigma})^5$ to $L^{2,\sigma}$. Our first theorem here improves this by showing that $\mathcal{T}_6$ in fact maps $(L^{2,\sigma})^5$ to $L^{2,\sigma+\delta}$ for some $\delta > 0$. By Fourier invariance, $\mathcal{T}_6$ maps $(H^\sigma)^5$ to $H^{\sigma+\delta}$.

The second result here establishes boundedness of $\mathcal{T}_6$ from $(L^{\infty,s})^5$ to $L^{\infty,s}$ for any $s > 1/2$. The analogous result for $s < 1/2$ is false, by scaling. We conclude by discussing the important borderline case $s = 1/2$.

Theorem 1.4.17. *For any $\sigma > 0$, $\mathcal{T}_6$ is bounded from $(L^{2,\sigma})^5$ to $L^{2,\sigma+\delta}$ with $\delta = \sigma/(1+\sigma) > 0$.*

Proof. By duality we need to prove that for all $f_1, \ldots, f_5, g \in L^2$ with $\|f_k\|_{L^2} = \|g\|_{L^2} = 1$, we have,

$$\left\langle \mathcal{T}_6\left(\langle x\rangle^{-\sigma} f_1, \ldots, \langle x\rangle^{-\sigma} f_5\right), \langle x\rangle^{\sigma+\delta} g\right\rangle_{L^2} \lesssim 1.$$

Unpacking this using (1.4.19), we see that this is the same as,

$$\begin{aligned}\int_{-1}^1 \int_{\mathbb{R}^3} & K(x,y,z,\lambda) f_1(z-y+x) f_2(\lambda z + x) f_3(\lambda y - y + x) \\ & \overline{f}_4(\lambda z - y + x)\overline{f}_5(z + \lambda y - y + x)\overline{g}(x)dydzdxd\lambda \lesssim 1,\end{aligned} \tag{1.4.45}$$

where,

$$K(x, y, z, \lambda) = \frac{\langle x \rangle^{\sigma+\delta}}{\langle z - y + x \rangle^{\sigma} \langle \lambda z + x \rangle^{\sigma} \langle \lambda y - y + x \rangle^{\sigma} \langle \lambda z - y + x \rangle^{\sigma} \langle z + \lambda y - y + x \rangle^{\sigma}}.$$

(The integration in λ here is over $[-1, 1]$. The integral in λ for $(-\infty, -1) \cup (1, \infty)$ can be transformed into this integral by the change of variables $\lambda \mapsto 1/\lambda$.)

The overall strategy is to identify a large set on which K is bounded, where controlling the integral is easy, and use a dyadic decomposition and finer bounds on $\mathcal{T}_6$ to control the integral on the set where K is not bounded. On the set where K is bounded the boundedness property,

$$\langle \mathcal{T}_6(f_1, f_2, f_3, f_4, f_5), g \rangle \lesssim \left(\prod_{k=1}^{5} \| f_k \|_{L^2}^5 \right) \| g \|_{L^2}.$$

deals with the integral automatically. So we only need to worry about the set where K is unbounded. On the unbounded piece the refined estimate,

$$\left\langle (\mathcal{T}_6)_{|\lambda - a| < \epsilon}(f, ..., f), g \right\rangle \le \epsilon \left(\prod_{k=1}^{5} \| f_k \|_{L^2}^5 \right) \| g \|_{L^2}.$$

will be used to gain control. This refined bound is a clear consequence of representation (1.4.20).

Fix ϵ small. We observe first that if $\epsilon|x| \le 1$ then $K \lesssim 1$. So we assume that $\epsilon|x| \ge 1$.

Now the relation,

$$|z - y + x|^2 + |\lambda z + x|^2 + |\lambda y - y + x|^2 = |\lambda z - y + x|^2 + |z + \lambda y - y + x|^2 + |x|^2,$$

gives that,

$$|x| \le \max\{|z - y + x|, |\lambda z + x|, |\lambda y - y + x|\},$$

and hence,

$$K \le \frac{\langle x \rangle^{\delta}}{\langle \lambda z - y + x \rangle^{\sigma} \langle z + \lambda y - y + x \rangle^{\sigma}}.$$

Now if,

$$|\lambda z - y + x| \ge \epsilon|x| \;\text{ or }\; |z + \lambda y - y + x| \ge \epsilon|x|,$$

then we automatically get $K \lesssim 1$ as $\delta \le \sigma$. Hence we assume that,

$$|\lambda z - y + x| \le \epsilon|x| \;\text{ and }\; |z + \lambda y - y + x| \le \epsilon|x|.$$

We now make some observations about how the sizes of $|y|$ and $|z|$ affect K. There are four cases.

1. First, suppose $|z|$ is large or comparable to $|x|$, so $|z| \ge 2\epsilon|x|$. Then,

$$2\epsilon|x| \le |z| \le |\lambda y - y + x| + |z + \lambda y - y + x| \le |\lambda y - y + x| + \epsilon|x|,$$

and hence $\epsilon|x| \le |\lambda y - y + x|$.

2. Next, suppose $|y|$ is large or comparable to $|x|$, so $|y| \geq 2\epsilon|x|$. Then,

$$2\epsilon|x| \leq |y| \leq |\lambda z + x| + |\lambda z - y + x| \leq |\lambda y - y + x| + \epsilon|x|,$$

and hence $\epsilon|x| \leq |\lambda z + x|$.

3. Next, suppose $|z|$ is small compared to $|x|$, so $|z| \leq 2\epsilon|x|$. Then,

$$|\lambda z + x| \geq |x| - |\lambda z| \geq |x| - |z| \geq (1 - 2\epsilon)|x|,$$

and so (by the smallness of ϵ) we have $\epsilon|x| \leq |\lambda z + x|$.

4. Finally, suppose $|y|$ is small compared to $|y|$, so $|y| \leq 2\epsilon|x|$. Then,

$$|\lambda y - y + x| \geq |x| - |(\lambda - 1)y| \geq |x| - 2|y| \geq (1 - 4\epsilon)|x|,$$

and so (by the smallness of ϵ) we have $\epsilon|x| \leq |\lambda y - y + x|$.

From these observations we see that if $|z|$ and $|y|$ are both large we get,

$$\langle x\rangle^{\sigma+\delta} \lesssim \langle \lambda z + x\rangle^{\sigma} \langle \lambda y - y + x\rangle^{\sigma},$$

and hence $K \lesssim 1$. If $|z|$ and $|y|$ are both small then we have the same bound on $\langle x\rangle$ and the same conclusion.

There are thus two regimes to consider: when $|y|$ is large and $|z|$ small, and when $|z|$ is large and $|y|$ small. For these regimes we will use a dyadic decomposition:

$$\langle x\rangle \sim 2^j, \quad \langle \lambda z - y + x\rangle \sim 2^k \text{ and } \langle z + \lambda y - y + x\rangle \sim 2^l.$$

Regime One: $|y|$ large, $|z|$ small.

From the observations above we have that $|\lambda z + x| \gtrsim \epsilon 2^j$. We have,

$$|z - y + x| \geq |\lambda y| - |z + \lambda y - y + x| \geq \frac{\epsilon}{2}|\lambda x| \gtrsim |\lambda| 2^j,$$

if we assume in addition that $|\lambda| 2^j \gtrsim \epsilon|\lambda||x|/2 \geq |z + \lambda y - y + x| = 2^l$. Under this assumption we have,

$$K \lesssim \frac{(2^j)^{\sigma+\delta}}{\langle 2^l\rangle \langle |\lambda| 2^j\rangle \langle 2^j\rangle} \lesssim 2^{j(\delta-\sigma)} 2^{-l\sigma} |\lambda|^{-\sigma},$$

which is bounded if $2^{j(\delta/\sigma-1)}2^{-l} \lesssim |\lambda|$. Hence in Regime One, K is bounded unless,

$$|\lambda| \leq \max\left\{2^{l-j}, 2^{j(\delta/\sigma-1)}2^{-l}\right\} =: \alpha.$$

By the bound,

$$K \lesssim \frac{|2^j|^{\sigma+\delta}}{|2^k|^{\sigma}|2^l|^{\sigma}|2^j|^{\sigma}} = \frac{2^{j\delta}}{2^{k\sigma}2^{l\sigma}},$$

we then have,

$$\mathcal{T}_6 \text{ on Regime One set} \lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^j \geq 2^k} \sum_{\epsilon 2^j \geq 2^l} \frac{2^{j\delta}}{2^{k\sigma} 2^{l\sigma}} \langle (\mathcal{T}_6)_{|\lambda| \leq \alpha}(f_1, f_2, f_3, f_4, f_5), g \rangle$$
$$\lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^j \geq 2^k} \sum_{\epsilon 2^j \geq 2^l} \frac{2^{j\delta}}{2^{k\sigma} 2^{l\sigma}} \max \left\{ 2^{l-j}, 2^{j(\delta/\sigma - 1)} 2^{-l} \right\} \lesssim 1.$$

Regime Two: $|z|$ large, $|y|$ small.

From the observations above we have that $|\lambda y - y + x| \gtrsim \epsilon 2^j$.

We have,

$$|z - y + x| \geq |(1-\lambda)z| - |\lambda z - y + x| \geq \frac{\epsilon}{2} |(1-\lambda)x| \gtrsim |1-\lambda| 2^j,$$

if we assume in addition that $|1-\lambda|2^j \lesssim \epsilon |1-\lambda||x|/2 \geq |\lambda z - y + x| = 2^k$. Under this assumptions we have,

$$K \lesssim \frac{(2^j)^{\sigma+\delta}}{\langle 2^k \rangle \langle |1-\lambda| 2^j \rangle \langle 2^j \rangle} = 2^{j(\delta-\sigma)} 2^{-k\sigma} |1-\lambda|^{-\sigma},$$

which is bounded if $2^{j(\delta/\sigma - 1)} 2^{-k} \lesssim |1-\lambda|$. Hence in Regime Two, K is bounded unless,

$$|1-\lambda| \leq \max \left\{ 2^{k-j}, 2^{j(\delta/\sigma-1)} 2^{-k} \right\} =: \alpha.$$

By the bound,

$$K \lesssim \frac{|2^j|^{\sigma+\delta}}{|2^k|^\sigma |2^l|^\sigma |2^j|^\sigma} = \frac{2^{j\delta}}{2^{k\sigma} 2^{l\sigma}},$$

we then have,

$$\mathcal{T}_6 \text{ on Regime Two set} \lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^j \geq 2^k} \sum_{\epsilon 2^j \geq 2^l} \frac{2^{j\delta}}{2^{k\sigma} 2^{l\sigma}} \langle \mathcal{T}_{|1-\lambda| \leq \alpha}(f, \ldots, f), g \rangle$$
$$\lesssim \sum_{\epsilon 2^j \geq 1} \sum_{\epsilon 2^j \geq 2^k} \sum_{\epsilon 2^j \geq 2^l} \frac{2^{j\delta}}{2^{k\sigma} 2^{l\sigma}} \max \left\{ 2^{k-j}, 2^{j(\delta/\sigma-1)} 2^{-k} \right\} \lesssim 1.$$

The bound (1.4.45) is thus established. □

Theorem 1.4.18. *For all $s > 1/2$, there is a constant C such that,*

$$\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{\infty,s}} \leq C \prod_{k=1}^{5} \|f_k\|_{L^{\infty,s}}. \tag{1.4.46}$$

Lemma 1.4.19. *Suppose that $v_1^2 + v_2^2 + v_3^2 = 1$. Then,*

$$\langle v_1 y_1 + v_2 y_2 + v_3 y_3 \rangle \leq \sqrt{2} \left[|v_1| \langle y_1 \rangle + |v_2| \langle y_2 \rangle + |v_3| \langle y_3 \rangle \right]. \tag{1.4.47}$$

Proof. We have,

$$
\begin{aligned}
1 + (v_1 y_1 + v_2 y_2 + v_3 y_3)^2 &\le 2\left(1 + v_1^2 y_1^2 + v_2^2 y_2^2 + v_3^2 y_3^2\right) \\
&= 2\left(v_1^2(1+y_1^2) + v_2^2(1+y_2^2) + v_3^2(1+y_3^2)\right) \\
&\le 2\left(|v_1|(1+y_1^2)^{1/2} + |v_2|(1+y_2^2)^{1/2} + |v_3|(1+y_3^2)^{1/2}\right)^2 .
\end{aligned}
$$

Taking square roots then gives (1.4.47) □

Proof of Theorem 1.4.18. We may assume by rescaling that $\|f_k\|_{L^{\infty,s}} = 1$, which means $|f_k(t)| \le \langle t\rangle^{-s}$. Set $y_k = (A(\lambda)x)_k$. Then, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$
\begin{aligned}
\|\mathcal{T}_6(f_1, f_2, f_3, f_4, f_5)\|_{L^{\infty,s}} &\le \sup_{x_1\in\mathbb{R}} \left(\langle x_1\rangle^s \mathcal{T}_6\left(\langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s}\right)(x_1)\right) \\
&= \sup_{x_1\in\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{1}{\lambda^2-\lambda+1} \frac{\langle x_1\rangle^s}{\langle y_1\rangle^s\langle y_2\rangle^s\langle y_3\rangle^s\langle x_2\rangle^s\langle x_3\rangle^s} dx_2 dx_3 d\lambda.
\end{aligned}
$$

We will show that the integral of the Japanese bracket terms over x_2 and x_3 can be bounded by an absolute constant independent of λ. Because $(\lambda^2-\lambda+1)^{-1}$ is integrable over $\mathbb{R}$, this will prove the bound.

By Cauchy–Schwarz, we have,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\langle x_1\rangle^s}{\langle y_1\rangle^s\langle y_2\rangle^s\langle y_3\rangle^s\langle x_2\rangle^s\langle x_3\rangle^s} dx_2 dx_3 d\lambda \\
&\qquad \le \left(\int_{\mathbb{R}^2} \frac{\langle x_1\rangle^{2s}}{\langle y_1\rangle^{2s}\langle y_2\rangle^{2s}\langle y_3\rangle^{2s}} dx_2 dx_3\right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{1}{\langle x_2\rangle^{2s}\langle x_3\rangle^{2s}} dx_2 dx_3\right)^{1/2} .
\end{aligned}
\tag{1.4.48}
$$

The second integral here splits as $\int_{\mathbb{R}} \langle x_2\rangle^{-2s} dx_2 \int_{\mathbb{R}} \langle x_3\rangle^{-2s} dx_3$, and is thus finite as $s > 1/2$.

To bound the first integral we must use some structure of $A(\lambda)$, which is given by,

$$
A(\lambda) = \frac{1}{\lambda^2-\lambda+1}\begin{pmatrix} \lambda & 1-\lambda & \lambda^2-\lambda \\ \lambda^2-\lambda & \lambda & 1-\lambda \\ 1-\lambda & \lambda^2-\lambda & \lambda \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.
$$

First we observe that the matrix has the following property. If we fix a row k and column j, then the determinent of the matrix obtained by deleting row k and column j is precisely a_{kj} – the element in row k and column j. This means that,

$$
a_{11} = \frac{\lambda}{\lambda^2-\lambda+1} = \det\left[\frac{1}{\lambda^2-\lambda+1}\begin{pmatrix} \lambda & 1-\lambda \\ \lambda^2-\lambda & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}, \tag{1.4.49}
$$

with similar formulas for a_{21} and a_{31}.

Next, because A is an isometry, the inverse matrix is just the transpose. Since $x = A^{-1}y$ we have the formula,

$$
x_1 = a_{11}y_1 + a_{21}y_2 + a_{31}y_3.
$$

A is an isometry, so $a_{11}^2 + a_{21}^2 + a_{31}^2 = 1$. We can therefore use (1.4.47), from the previous lemma, raised to to the power s; it reads,

$$
\langle x_1\rangle^{2s} \lesssim |a_{11}|^{2s}\langle y_1\rangle^{2s} + |a_{21}|^{2s}\langle y_2\rangle^{2s} + |a_{31}|^{2s}\langle y_3\rangle^{2s}.
$$

Applying this to bound the first integral in (1.4.48), we then have,

$$\int_{\mathbb{R}^2} \frac{\langle x_1\rangle^{2s}}{\langle y_1\rangle^{2s}\langle y_2\rangle^{2s}\langle y_3\rangle^{2s}} dx_2 dx_3$$
$$\lesssim \int_{\mathbb{R}^2} \frac{|a_{11}|}{\langle y_2\rangle^{2s}\langle y_3\rangle^{2s}} dx_2 dx_3 + \int_{\mathbb{R}^2} \frac{|a_{21}|}{\langle y_1\rangle^{2s}\langle y_3\rangle^{2s}} dx_2 dx_3 + \int_{\mathbb{R}^2} \frac{|a_{31}|}{\langle y_1\rangle^{2s}\langle y_2\rangle^{2s}} dx_2 dx_3.$$

We will show how the first integral may be bounded; the other two are bounded by an identical argument. We perform the change of variables $z_2 = y_2 = (Ax)_2$ and $z_3 = y_3 = (Ax)_3$. Expressed as a matrix, this change of variables is,

$$\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}.$$

The determinent of this change of variables is, by (1.4.49), simply $|a_{11}|$. Therefore, using that $|a_{11}| \leq 1$ because A is an isometry,

$$\int_{\mathbb{R}^2} \frac{|a_{11}|^{2s}}{\langle y_2\rangle^{2s}\langle y_3\rangle^{2s}} dx_2 dx_3 = \int_{\mathbb{R}^2} \frac{|a_{11}|^{2s-1}}{\langle z_2\rangle^{2s}\langle z_3\rangle^{2s}} dz_2 dz_3 \leq \int_{\mathbb{R}} \frac{1}{\langle z_2\rangle^{2s}} dz_2 \int_{\mathbb{R}} \frac{1}{\langle z_3\rangle^{2s}} dz_3,$$

and the right hand side is finite because $s > 1/2$. □

Proposition 1.4.20. *If $s < 1/2$, there is no constant C such that $\|\mathcal{T}_6(f,f,f,f,f)\|_{L^{\infty,s}} \leq C\|f\|^5_{L^{\infty,s}}$ for all functions $f \in L^{\infty,s}$.*

The proof of the proposition involves a standard scaling argument, which we omit.

The index $s = 1/2$, which is the borderline between Theorem 1.4.18 and Proposition 1.4.20, is especially relevant because the space $\dot{L}^{\infty,1/2}$ is a critical space for the equation $iu_t = \mathcal{T}_6(u,u,u,u,u)$. That is, both the equation and the space are invariant under the scaling $f(x) \mapsto \lambda^{1/2} f(\lambda x)$. By analogy with the cubic continuous resonant equation in dimension two, it seems reasonable to conjecture that the operator $\mathcal{T}_6$ is bounded from $(\dot{L}^{\infty,1/2})^5$ to $\dot{L}^{\infty,1/2}$. In fact, this is equivalent to the statement that,

$$\sup_{x\in\mathbb{R}} \mathcal{T}_6\left(\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}\right)(x)\sqrt{x} < \infty,$$

which by scaling is equivalent to,

$$w = \mathcal{T}_6\left(\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}\right)(1) < \infty,$$

and again by scaling, this is equivalent to,

$$\frac{w}{\sqrt{x}} = \mathcal{T}_6\left(\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}\right)(x),$$

for some $|w| < \infty$. In all, boundedness from $(\dot{L}^{\infty,1/2})^5$ to $\dot{L}^{\infty,1/2}$ is equivalent to $1/\sqrt{x}$ being a stationary wave. In [19] it is proved that $1/x$ is a stationary wave of the analogous continuous resonant system. The proof, unfortunately, does not extend to the present situation in a manageable way.

Finally, let us note that $1/\sqrt{x} \in L^{\infty,1/2} = L^{2,\infty}$ being a stationary wave would show that Corollary 1.4.16 is sharp. That result states that once a stationary wave is in L^2, it is analytic and exponentially decaying in physical and Fourier space. The function $1/\sqrt{x}$ being a stationary wave would show that the result cannot be strengthened even to weak L^2.

§1.5 · THE CUBIC RESONANT EQUATION

In this final section we study the system defined by the Hamiltonian,

$$\mathcal{H}_4(f) = \frac{2}{\pi}\|e^{itH} f\|^4_{L^4_t L^4_x} = \frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}} |e^{itH} f(x)|^4 dx dt, \tag{1.5.1}$$

which has an associated multilinear functional,

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}} (e^{itH} f_1(x))(e^{itH} f_2(x))\overline{(e^{itH} f_3(x))(e^{itH} f_4(x))} dx dt. \tag{1.5.2}$$

The functional has a large number of permutation symmetries,

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \mathcal{E}_4(f_2, f_1, f_3, f_4) = \mathcal{E}_4(f_1, f_2, f_4, f_3) = \overline{\mathcal{E}_4(f_3, f_4, f_1, f_2)}. \tag{1.5.3}$$

As in the case of quintic resonant system, Hamilton's equation is $iu_t = \mathcal{T}_4(u, u, u)$ where $\mathcal{T}_4$ is defined by,

$$\langle \mathcal{T}_4(f_1, f_2, f_3), g\rangle = 4\mathcal{E}_4(f_1, f_2, f_3, g). \tag{1.5.4}$$

By an identical computation to the derivation of (1.4.7), we find that the operator $\mathcal{T}_4$ is given explictly by,

$$\mathcal{T}_4(f_1, f_2, f_3)(x) = \frac{8}{\pi}\int_{-\pi/4}^{\pi/4} e^{-itH}\left[(e^{itH} f_1)(e^{itH} f_2)\overline{(e^{itH} f_3)}\right](x) dt. \tag{1.5.5}$$

This shows that the flow corresponding to the Hamiltonian $\mathcal{H}_4$ is precisely the resonant equation (1.2.9) in the case $k = 1$. As discussed in the introduction, it was shown in [32] that this system is also the modified scattering limit of the NLS equation (1.1.6).

1.5.1 · REPRESENTATIONS OF THE HAMILTONIAN AND THE FLOW OPERATOR

As for the quintic case, we devote a significant amount of work to determining alternative representations of $\mathcal{E}_4$, $\mathcal{H}_4$ and $\mathcal{T}_4$. In contrast to the quintic case, we do *not* have representations for $\mathcal{H}_4$ of the form,

$$\int_{\mathbb{R}^4} f_1(y_1) f_2(y_2)\overline{f_3(y_3) f_4(y_4)}\delta_{y_1+y_2=y_3+y_4}\delta_{y_1^2+y_2^2=y_3^2+y_4^2} dy \quad \text{or} \quad \|e^{it\Delta} f\|^4_{L^4_t L^4_x}.$$

These representations are inconsistent with the scaling of the inequality $\mathcal{H}_4(f) \le (1/\sqrt{8\pi})\|f\|^4_{L^2}$ which we prove in Theorem 1.5.7.

Theorem 1.5.1. *There holds the representations,*

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{1}{2\pi^2}\int_{\mathbb{R}^4} e^{-\frac{1}{2}[(\lambda v_2)^2+v_1^2]} f_1(\lambda v_1 + v_3) f_2(v_2+v_3) \tag{1.5.6}$$

$$\overline{f_3(\lambda v_1 + v_2 + v_3) f_4(v_3)} dv_1 dv_2 dv_3 d\lambda, \tag{1.5.7}$$

$$\mathcal{T}_4(f_1, f_2, f_3)(x) = \frac{2}{\pi^2}\int_{\mathbb{R}^4} e^{-\frac{1}{2}[(\lambda v_2)^2+v_1^2]} f_1(\lambda v_1 + x) f_2(v_2 + x)\overline{f_3(\lambda v_1 + v_2 + x)} dv_1 dv_2 d\lambda. \tag{1.5.8}$$

To prove this theorem we need a lemma.

Lemma 1.5.2. *Let $\psi(x) = (1+x^2)^{-1/2}$. Then $\hat{\psi}(\xi) = \zeta(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{1}{|v|} e^{-\frac{1}{2}[(\xi/v)^2+v^2]} dv$.*

Proof. It is clear that ψ is in L^2. We will calculate the Fourier transform of $\zeta(\xi)$ and find that it equals ψ. The lemma then follows from Fourier inversion and the fact that ψ is even.

We have,

$$\begin{aligned}\hat{\zeta}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\int_{\mathbb{R}} \frac{1}{|v|} e^{-\frac{1}{2}[(\xi/v)^2+v^2]} dv d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{|v|} e^{-\frac{1}{2}v^2}\int_{\mathbb{R}} e^{ix\xi} e^{-\frac{1}{2}(\xi/v)^2} d\xi dv = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}v^2} e^{-\frac{1}{2}v^2x^2} dv,\end{aligned}$$

where in the last equality we used the explicit Fourier transform of the Gaussian e^{-ax^2} with $a = 1/(2v^2)$. In this last integral we perform the change of variables $u = v(1+x^2)^{-1/2}$, which gives,

$$\hat{\zeta}(x) = \frac{1}{(1+x^2)^{1/2}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}u^2} du = \frac{1}{(1+x^2)^{1/2}} = \psi(x),$$

which is what we wanted to prove. □

Proof of Theorem 1.5.1. We evaluate (1.5.2) using the Mehler formula (1.2.2), which reads,

$$e^{itH} f_k(x) = \frac{1}{\sqrt{2\pi}|\sin(2t)|^{1/2}}\int_{\mathbb{R}} e^{-i\frac{(x^2/2+y^2/2)\cos(2t)-xy}{\sin(2t)}} f_k(y) dy.$$

For notational convenience, let $\Lambda(x,t) = (e^{itH} f_1)(e^{itH} f_2)\overline{(e^{itH} f_3)(e^{itH} f_4)}$ be the integrand in (1.5.2). Using the Mehler formula, we have,

$$\Lambda(x,t) = \frac{1}{4\pi^2|\sin(2t)|^2}\int_{\mathbb{R}^4} e^{-i\frac{\Omega\cos(2t)}{2\sin(2t)}} e^{-i\frac{(y_1+y_2-y_3-y_4)x}{\sin(2t)}} f_1(y_1) f_2(y_2)\overline{f_3(y_3) f_4(y_4)} dy_1 dy_2 dy_3 dy_4$$

where $\Omega = y_1^2 + y_2^2 - y_3^2 - y_4^2$. Changing variables $w(y_3) = -y_1 - y_2 + y_3 + y_4$ and integrating over x yields,

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda(x,t)dx \\
&= \frac{1}{4\pi^2|\sin(2t)|^2} \int_{\mathbb{R}} \int_{\mathbb{R}^4} e^{-i\frac{\Omega\cos(2t)}{2\sin(2t)}} e^{i\frac{wx}{\sin(2t)}} f_1(y_1) f_2(y_2) \\
&\qquad \overline{f_3(w + y_1 + y_2 - y_4) f_4(y_4)} dw dy_1 dy_2 dy_4 dx \\
&= \frac{1}{2\pi|\sin(2t)|} \int_{\mathbb{R}^3} e^{-i\frac{\Omega\cos(2t)}{2\sin(2t)}} f_1(y_1) f_2(y_2) \overline{f_3(y_1 + y_2 - y_4) f_4(y_4)} dy_2 dy_3 dy_4,
\end{aligned}$$

where to get the second equality we used the Fourier inversion formula (1.1.18) with $a = 1/\sin(2t)$.

We now integrate t on the interval $[-\pi/4, \pi/4]$ and then perform the change of variables $u = -\cos(2t)/\sin(2t)$. This change of variables bijectively maps $(-\pi/4, 0) \cup (0, \pi/4]$ to $(-\infty, +\infty)$ and satisfies $du = 2dt/\sin^2(2t)$. Moreover,

$$u^2 = \frac{\cos^2(2t)}{\sin^2(2t)} = \frac{1}{\sin^2(2t)} - 1,$$

which gives $\sin(2t) = (u^2 + 1)^{-1/2}$. Using these, we find,

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} \Lambda(x,t) dx dt \\
&\qquad = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}} e^{-i\frac{\Omega}{2}u} \frac{1}{(1+u^2)^{1/2}} du \right) f_1(y_1) f_2(y_2) \overline{f_3(y_1 + y_2 - y_4) f_4(y_4)} dy_2 dy_3 dy_4.
\end{aligned}$$

We notice that the term inside round parentheses is the Fourier transform of $\sqrt{2\pi}(1 + u^2)^{-1/2}$ evaluated at the point $-\Omega/2$. Therefore, by Lemma 1.5.2, we have,

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4} \int_{\mathbb{R}} \Lambda(x,t) dx dt \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}} \frac{1}{|v_1|} e^{-\frac{1}{2}[(\Omega/2v_1)^2 + v_1^2]} dv_1 \right) f_1(y_1) f_2(y_2) \overline{f_3(y_1 + y_2 - y_4) f_4(y_4)} dy_2 dy_3 dy_4.
\end{aligned}$$

At this point $\Omega/2 = [(y_1)^2 + (y_2)^2 - (y_1 + y_2 - y_4)^2 - (y_4)^2]/2 = (y_1 - y_4)(y_2 - y_4)$. For fixed v_1, we perform the linear change of variables,

$$\begin{pmatrix} \lambda \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (y_1 - y_4)/v_1 \\ y_2 - y_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1/v_1 & 0 & -1/v_1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix},$$

which has determinant $1/|v_1|$. The inverse is given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} \lambda v_1 + v_3 \\ v_2 + v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ v_2 \\ v_3 \end{pmatrix},$$

and we note specifically that $\Omega/2v_1 = [(y_1 - y_4)/v_1](y_2 - y_4) = \lambda v_2$. This gives

$$\int_{-\pi/4}^{\pi/4}\int_{\mathbb{R}} \Lambda(x,t)dxdt = \frac{1}{4\pi}\int_{\mathbb{R}^4} e^{-\frac{1}{2}[(\lambda v_2)^2+v_1^2]} f_1(\lambda v_1 + v_3) f_2(v_2+v_3) \overline{f_3(\lambda v_1 + v_2 + v_3) f_4(v_3)} dv_1 dv_2 dv_3 d\lambda,$$

which is formula (1.5.7).

To get (1.5.8), we simply use the relation $\langle \mathcal{T}_4(f_1, f_2, f_3), g\rangle = 4\mathcal{E}_4(f_1, f_2, f_3, f_4)$. □

Theorem 1.5.3. *Let $G(x) = e^{-\frac{1}{2}x^2}$. There holds the representations,*

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2}\pi^2}\int_{\mathbb{R}} \frac{1}{1+\lambda^2} E_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4) d\lambda, \tag{1.5.9}$$

$$\mathcal{T}_4(f_1, f_2, f_3) = \frac{1}{\sqrt{2}\pi^2}\int_{\mathbb{R}} \frac{1}{1+\lambda^2} T_{B(\lambda)}(G, f_1, f_2, G, f_3) d\lambda, \tag{1.5.10}$$

where for every λ, $B(\lambda)$ is an isometry and $B(\lambda)(0,1,1) = (0,1,1)$.

Proof. We observe that,

$$(\lambda v_2)^2 + (v_1)^2 = \left(\frac{\lambda v_2 - v_1}{\sqrt{2}}\right)^2 + \left(\frac{\lambda v_2 + v_1}{\sqrt{2}}\right)^2,$$

which gives,

$$e^{-\frac{1}{2}[(\lambda v_2)^2 + (v_1)^2]} = G\left(\frac{\lambda v_2 - v_1}{\sqrt{2}}\right) G\left(\frac{\lambda v_2 + v_1}{\sqrt{2}}\right).$$

We substitute this expression into (1.5.7). Using the fact that $\overline{G(x)} = G(x)$, this gives,

$$\begin{aligned}&\mathcal{E}_4(f_1, f_2, f_3, f_4)\\ &= \frac{1}{2\pi^2}\int_{\mathbb{R}^4} G\left(\frac{\lambda v_2 + v_1}{\sqrt{2}}\right) f(\lambda v_1 + v_3) f(v_2 + v_3) \\ &\qquad \overline{G\left(\frac{\lambda v_2 - v_1}{\sqrt{2}}\right)}\, \overline{f(\lambda v_1 + v_2 + v_3) f(v_3)} dv_1 dv_2 dv_3.\end{aligned}$$

By looking at the arguments of the functions in the integrand, we are led to define the matrices $C(\lambda)$ and $D(\lambda)$ by,

$$C(\lambda)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & \lambda/\sqrt{2} & 0 \\ \lambda & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (\lambda v_2 + v_1)/\sqrt{2} \\ \lambda v_1 + v_3 \\ v_2 + v_3 \end{pmatrix},$$

and,

$$D(\lambda)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & \lambda/\sqrt{2} & 0 \\ \lambda & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (\lambda v_2 - v_1)/\sqrt{2} \\ \lambda v_1 + v_2 + v_3 \\ v_3 \end{pmatrix}.$$

We perform the change of variables $w = D(\lambda)v$. We determine that $\det D(\lambda) = (1+\lambda^2)/\sqrt{2} > 0$, and so $v = D(\lambda)^{-1}w$ is well defined. Set $B(\lambda) = C(\lambda)D(\lambda)^{-1}$. Performing the change of variables then yields the expression (1.5.9). The equation for $\mathcal{T}_4$ follows from (1.5.4). A calculation reveals that $B(\lambda)$ is given explicitly by

$$B(\lambda) = C(\lambda)D(\lambda)^{-1} = \frac{1}{1+\lambda^2}\begin{pmatrix} -1+\lambda^2 & \lambda\sqrt{2} & -\lambda\sqrt{2} \\ -\lambda\sqrt{2} & \lambda^2 & 1 \\ \lambda\sqrt{2} & 1 & \lambda^2 \end{pmatrix}. \tag{1.5.11}$$

We finally verify the two properties of $B(\lambda)$.

1. The relationship,

$$\left(\frac{\lambda v_2 + v_1}{\sqrt{2}}\right)^2 + (\lambda v_1 + v_3)^2 + (v_2 + v_3)^2 = \left(\frac{\lambda v_2 - v_1}{\sqrt{2}}\right)^2 + (\lambda v_1 + v_2 + v_3)^2 + (v_3)^2,$$

means that for all $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ we have $|C(\lambda)v|^2 = |D(\lambda)v|^2$. Replacing v with $D(\lambda)^{-1}v$ gives $|B(\lambda)v| = |v|$, so that $B(\lambda)$ is an isometry.

2. Set $e = (0, 1, 1)$. The relationship among the arguments of the functions f_k,

$$(\lambda v_1 + v_3) + (v_2 + v_3) = (\lambda v_1 + v_2 + v_3) + (v_3),$$

means that for all $v \in \mathbb{R}^3$, $\langle C(\lambda)v, e\rangle = \langle D(\lambda)v, e\rangle$. Replacing v by $D(\lambda)^{-1}v$, we have $\langle v, e\rangle = \langle B(\lambda)v, e\rangle = \langle v, B(\lambda)^{-1}e\rangle$, which means $e = B(\lambda)^{-1}e$ and hence $B(\lambda)e = e$. ☐

Theorem 1.5.4. *There holds the representations,*

$$\mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} E_{S(\theta)}(G, f_1, f_2, G, f_3, f_4)d\theta, \tag{1.5.12}$$

$$\mathcal{T}_4(f_1, f_2, f_3)(x) = \frac{\sqrt{2}}{\pi^2}\int_0^{2\pi} T_{S(\theta)}(G, f_1, f_2, G, f_3)(x)d\theta, \tag{1.5.13}$$

where $S(\theta)$ is the rotation of $\mathbb{R}^3$ by θ radians about the axis $(0, 1, 1)$.

Proof. Because the matrix $B(\lambda)$ is an isometry, $\det(B(\lambda)) = +1$, and $B(\lambda)(0, 1, 1) = (0, 1, 1)$, it must, in fact, be a rotation about the axis $(0, 1, 1)$.

For any rotation A of $\mathbb{R}^3$, the angle of rotation θ satisfies, $2\cos(\theta) + 1 = \mathrm{Trace}(A)$. In the present case, this means,

$$\cos(\theta) = \phi(\lambda) := \frac{1}{2}(\mathrm{Trace}(B(\lambda)) - 1) = \frac{1}{2}\left(\frac{1-3\lambda^2}{1+\lambda^2} - 1\right). \tag{1.5.14}$$

We carefully perform the change of variables $\lambda \mapsto \theta$ in (1.5.9).

We find that $\phi(0) = 1$, that ϕ is increasing on $(-\infty, 0]$, decreasing on $[0, +\infty)$, and that as $\lambda \to \pm\infty$, $\phi(\lambda) \to -1$. By setting $\lambda = 1$ in (1.5.11), we find that $\sin(\theta) = -1 < 0$, and hence $\theta = -3\pi/2 \in [\pi, 2\pi]$. By setting $\lambda = -1$ in (1.5.11), we find that $\sin(\theta) = 1 > 0$, and hence $\theta = \pi/2 \in [0, \pi]$. From these considerations and continuity, we infer that under $\lambda \mapsto \theta$, $\mathbb{R}$ is bijectively mapped to $(0, 2\pi]$.

To perform the change of variables, we need to find its determinant. It is given by,

$$\begin{aligned} \frac{d\theta}{d\lambda} &= \left|\frac{d}{d\lambda}\arccos\left(\frac{1}{2}\left[\frac{-1+3\lambda^2}{\lambda^2+1} - 1\right]\right)\right| = \left|\frac{d}{d\lambda}\arccos\left(\frac{\lambda^2-1}{\lambda^2+1}\right)\right| \\ &= \left|\frac{d}{d\lambda}\arctan\left(\frac{2\lambda}{\lambda^2+1}\right)\right| = 2\left|\frac{d}{d\lambda}\arctan(\lambda)\right| = \frac{2}{1+\lambda^2}. \end{aligned}$$

Formula (1.5.12) then follows. □

Formula (1.5.12) is very similar to the formula (1.4.26) derived in Section 4,

$$\mathcal{E}_6(f_1, f_2, f_3, f_4, f_5, f_6) = \frac{1}{2\sqrt{3}\pi^2} \int_0^{2\pi} E_{R(\theta)}(f_1, f_2, f_3, f_4, f_5, f_6)\, d\theta, \tag{1.5.15}$$

where $R(\theta)$ is the rotation of $\mathbb{R}^3$ by θ radians about the axis $(1, 1, 1)$. In both cases the functionals $\mathcal{E}_6$ and $\mathcal{E}_4$ are expressed in terms of a family of rotations about a fixed axis. The specific axis in both cases is basically determined by the fact that both functionals are translation invariant.

The presence of G in (1.5.12) has concrete consequences for the symmetry structure of $\mathcal{E}_4$. While the functionals $E_{S(\theta)}$ are all invariant under L^2 scaling, for example, this is not inherited by $\mathcal{E}_4$ because the scaling transformation also affects the G terms, but these must be fixed. Similarly, the functional $\mathcal{E}_4$ is not invariant under the actions $f_k \mapsto e^{i\lambda x^2} f_k$ and $f_k \mapsto e^{i\lambda\Delta} f_k$. (However we will see that the functional is invariant under the action $f_k \mapsto e^{i\lambda H} f_k$ because G is a fixed point for this.) Finally, we will also see that the presence of the G terms has consequences for the set of saturating functions of the L^2 bound for $\mathcal{E}_4$.

1.5.2 · Symmetries of the Hamiltonian and conserved quantities of the flow

Theorem 1.5.5. *The function $\mathcal{E}_4(f_1, f_2, f_3, f_4)$ is invariant under the following actions:*

(i) Fourier transform, $f_k \mapsto \widehat{f}_k$.

(ii) Modulation, $f_k \mapsto e^{i\lambda} f_k$.

(iii) Linear modulation, $f_k \mapsto e^{i\lambda} f_k$.

(iv) Translation, $f_k \mapsto f_k(\cdot + \lambda)$.

(v) Schrödinger with harmonic trapping group, $f_k \mapsto e^{i\lambda H} f_k$.

Proof. Because $S(\theta)$ is an isometry for all θ, items (i) through (v) of Theorem 1.3.3 apply to the functional $E_{S(\theta)}$.

(i) Let $\mathcal{F}$ denote the Fourier transform. Note that $\mathcal{F}(G) = \mathcal{F}(e^{-x^2/2}) = e^{-x^2/2} = G$. Therefore,

$$\begin{aligned}\mathcal{E}_4(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4) &= \frac{1}{\sqrt{8}\pi^2} \int_0^{2\pi} \mathcal{F}\left(E_{S(\theta)}(G, \widehat{f}_1, \widehat{f}_2, G, \widehat{f}_3, \widehat{f}_4)\right) d\theta \\ &= \frac{1}{\sqrt{8}\pi^2} \int_0^{2\pi} E_{S(\theta)}(\widehat{G}, f_1, f_2, \widehat{G}, f_3, f_4)\, d\theta = \mathcal{E}_4(f_1, f_2, f_3, f_4).\end{aligned}$$

(ii) This is an easy consequence of representation (1.5.7).

(iii) This again follows from representation (1.5.7).

(iv) This follows from (iii), along with the Fourier transform symmetry in (i).

(v) Recall that $G(x) = e^{-x^2/2}$ is (a constant multiple of) the first Hermite function and hence satisfies $e^{itH}G(x) = e^{it}G(x)$. This then gives, for each θ and s,

$$\begin{aligned} E_{S(\theta)}(G, e^{isH}f_1, e^{isH}f_2, G, e^{isH}f_3, e^{isH}f_4) &= E_{S(\theta)}(e^{-isH}G, f_1, f_2, e^{-isH}G, f_3, f_4) \\ &= E_{S(\theta)}(e^{-is}G, f_1, f_2, e^{-is}G, f_3, f_4) \\ &= E_{S(\theta)}(G, e^{is}f_1, e^{is}f_2, G, e^{is}f_3, e^{is}f_4). \end{aligned}$$

Thus,

$$\mathcal{E}_4(e^{isH}f_1, e^{isH}f_2, e^{isH}f_3, e^{isH}f_4) = \mathcal{E}_4(e^{is}f_1, e^{is}f_2, e^{is}f_3, e^{is}f_4) = \mathcal{E}_4(f_1, f_2, f_3, f_4),$$

using (ii). □

Corollary 1.5.6. *We have the following commuter equalities,*

$$e^{i\lambda Q}\mathcal{T}_4(f_1, f_2, f_3) = \mathcal{T}_4(e^{i\lambda Q}f_1, e^{i\lambda Q}f_2, e^{i\lambda Q}f_3), \tag{1.5.16}$$

$$Q\mathcal{T}_4(f_1, f_2, f_3) = \mathcal{T}_4(Qf_1, f_2, f_3) + \mathcal{T}_4(f_1, Qf_2, f_3) - \mathcal{T}(f_1, f_2, Qf_3), \tag{1.5.17}$$

where Q are the operators: $Q = 1$, $Q = x$, $Q = id/dx$, and $Q = H$.

The corollary follows immediately from Theorem 1.3.4. By Noether's Theorem, Theorem 1.3.5, we determine four conserved quantities for the Hamiltonian flow corresponding to $\mathcal{H}_4$. These are summarized in the following table.

Symmetry of $\mathcal{H}_4$	Conserved quantity	Operator commuting with $\mathcal{T}_4$
$f \mapsto e^{i\lambda}f$	$\int_{\mathbb{R}} \lvert f(x)\rvert^2 dx$	1
$f \mapsto e^{i\lambda x}f$	$\int_{\mathbb{R}} x\lvert f(x)\rvert^2 dx$	x
$f \mapsto f(\cdot + \lambda)$	$\operatorname{Re} \int_{\mathbb{R}} f'(x)\overline{f}(x)dx$	d/dx
$f \mapsto e^{i\lambda\mathcal{H}}f$	$\int_{\mathbb{R}} \lvert xf(x)\rvert^2 + \lvert f'(x)\rvert^2 dx$	H

1.5.3 · Boundedness of the functional and wellposedness of Hamilton's equation

Proposition 1.5.7. *We have the following sharp bound,*

$$\lvert\mathcal{E}_4(f_1, f_2, f_3, f_4)\rvert \le \frac{1}{\sqrt{2\pi}} \prod_{k=1}^{4} \lVert f_k\rVert_{L^2}, \tag{1.5.18}$$

with equality if and only if the functions f_k are the same Gaussian $f_k(x) = e^{-\frac{1}{2}x^2+\beta x}$ for some $\beta \in \mathbb{C}$.

In particular there holds $\mathcal{H}_4(f) \le (1/\sqrt{2\pi})\lVert f\rVert_{L^2}^4$ with equality if and only if $f(x) = e^{-\frac{1}{2}x^2+\beta x}$ for some $\beta \in \mathbb{C}$.

The equality case here is a little different to the analogous result for $\mathcal{E}_6$ in Theorem 1.4.10. For $\mathcal{E}_6$, the set of saturating functions is all Gaussians of the form $e^{-\alpha x^2+\beta x}$ with $\operatorname{Re}\alpha > 0$. In the case of $\mathcal{E}_4$, we necessarily have $\alpha = 1/2$.

Proof. Using the representation (1.5.12), we find that,

$$|\mathcal{E}_4(f_1, f_2, f_3, f_4)| \leq \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} |E_{S(\theta)}(G, f_1, f_2, G, f_3, f_4)| d\theta, \leq \frac{1}{\sqrt{2}\pi}\|G\|_{L^2}^2 \prod_{k=1}^{4} \|f_k\|_{L^2},$$

We calculate $\|G\|_{L^2}^2 = \int_{\mathbb{R}} (e^{-x^2/2})^2 dx = \sqrt{\pi}$, which yields the inequality.

To have equality, we must have ,

$$|E_{S(\theta)}(G, f_1, f_2, G, f_3, f_4)| = \|G\|_{L^2}^2 \prod_{k=1}^{4} \|f_k\|_{L^2}, \tag{1.5.19}$$

for almost all θ. Because $S(\theta)$ is not a signed permutation almost everywhere, functions f_k satisfying (1.5.19) must necessarily be Gaussians by Theorem 1.3.12. To find which Gaussians are admissible, write,

$$f_k(x) = e^{-\frac{1}{2}x^2 + \alpha_k x^2 + \beta_k x}, \tag{1.5.20}$$

for $\alpha_k, \beta_k \in \mathbb{C}$. The equality condition from Theorem 1.3.12 reads, for all θ and all $x \in \mathbb{R}^3$,

$$G((S(\theta)x)_1) f_1((S(\theta)x)_2) f_2((S(\theta)x)_3) = G(x_1) f_3(x_2) f_4(x_3). \tag{1.5.21}$$

We substitute the expressions for f_k in (1.5.20) and $G(x) = e^{-x^2/2}$ into (1.5.21). Because $S(\theta)$ is an isometry, the terms involving $-x^2/2$ will cancel, leaving,

$$\alpha_1 (S(\theta)x)_2^2 + \beta_1 (S(\theta)x)_2 + \alpha_2 (S(\theta)x)_3^2 + \beta_2 (S(\theta)x)_3 = \alpha_3 x_2^2 + \beta_3 x_2 + \alpha_4 x_3^2 + \beta_4 x_3. \tag{1.5.22}$$

Set $x = (0, k, k)$. Then $S(\theta)x = x$ and so $(\alpha_1 + \alpha_2)k^2 + (\beta_1 + \beta_2)k = (\alpha_3 + \alpha_4)k^2 + (\beta_3 + \beta_4)k$. This gives $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ and $\beta_1 + \beta_2 = \beta_3 + \beta_4$.

Next set $x = (1, 0, 0)$. Then $S(\theta)x = (\cos(\theta), -\sin(\theta)/\sqrt{2}, \sin(\theta)/\sqrt{2})$, which gives,

$$(\alpha_1 + \alpha_2)\frac{\sin(\theta)^2}{2} + (\beta_1 - \beta_2)\frac{\sin(\theta)}{\sqrt{2}} = 0,$$

and hence $\alpha_1 = -\alpha_2$ and $\beta_1 = \beta_2$. Setting $x = S(-\theta)(1, 0, 0)$ similarly yields $\alpha_3 = -\alpha_4$ and $\beta_3 = \beta_4$. This gives $\beta_1 = \beta_2 = \beta_3 = \beta_4$. Because $S(\theta)(0, 1, 1) = (0, 1, 1)$, the β terms in (1.5.22) cancel, and so any β is admissible.

Finally set $x = (1, 1, 1) = (1, 0, 0) + (0, 1, 1)$. Then $S(\theta)x = (\cos(\theta), 1 - \sin(\theta)/\sqrt{2}, 1 + \sin(\theta)/\sqrt{2})$, and,

$$\alpha_1 \left(1 - \frac{\sin(\theta)}{\sqrt{2}}\right)^2 - \alpha_1 \left(1 + \frac{\sin(\theta)}{\sqrt{2}}\right)^2 = \alpha_3 + \alpha_4 = 0,$$

expanding the left hand side we find that $2\alpha_1 \sin(\theta) = 0$, and hence $\alpha_1 = \alpha_2 = 0$. Similarly by considering $x = S(-\theta)(1, 1, 1)$ we find that $\alpha_3 = \alpha_4 = 0$.

In conclusion, the functions f_k in (1.5.20) satisfy the equality condition (1.5.21) if and only if $\alpha_k = 0$ and $\beta_k = \beta_1$ for all k. □

Theorem 1.5.8. *We have the operator bound,*

$$\|\mathcal{T}_4(f_1, f_2, f_3)\|_X \le C_X \prod_{k=1}^{3} \|f_k\|_X,$$

for the spaces,

(i) $X = L^2$ *with* $C_X = \sqrt{8/\pi}$.

(ii) $X = L^{2,\sigma}$, *for any* $\sigma \ge 0$.

(iii) $X = H^\sigma$, *for any* $\sigma \ge 0$.

(iv) $X = L^{\infty,s}$, *for any* $s > 1/2$.

(v) $X = L^{p,s}$, *for any* $p \ge 2$ *and* $s > 1/2 - 1/p$.

Proof. The bounds (i) through (iii) follow from the representation (1.5.13) along with bounds on $T_{S(\theta)}$ from Theorem 1.3.7, noting that in all cases $\|G\|_X < \infty$.

For (iv), we need to show $\sup_{x\in\mathbb{R}} |\mathcal{T}_4(\langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s})(x)\langle x\rangle^s| < \infty$. As in the proof of Theorem 1.4.18, it is sufficient to show that,

$$\sup_{x\in\mathbb{R}} T_{B(\lambda)}(e^{-t^2/2}, \langle t\rangle^{-s}, \langle t\rangle^{-s}, e^{-t^2/2}, \langle t\rangle^{s})(x)\langle x\rangle^{-s} \le C,$$

for some C independent of λ. We observe that we have $e^{-t^2/2} \lesssim \langle t\rangle^{-s}$, which means it is sufficient to show that,

$$\sup_{x\in\mathbb{R}} T_{B(\lambda)}(\langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s}, \langle t\rangle^{-s})(x)\langle x\rangle^{s} \le C,$$

for some C independent of λ. The proof of this bound is identical to the proof of the analogous bound in Theorem 1.4.18.

Item (v) follows from interpolating between $L^{2,\sigma}$ and $L^{\infty,s}$. □

Theorem 1.5.9. *Consider the Cauchy problem,*

$$\begin{aligned} iu_t &= \mathcal{T}_4(u,u,u), \\ f(t=0) &= f_0, \end{aligned} \tag{1.5.23}$$

which is Hamilton's equation corresponding to $\mathcal{H}_4$ *and the resonant equation* (1.1.4) *in the cubic case* $k = 1$.

(i) The Cauchy problem (1.5.23) *is locally wellposed in* X *for any of the spaces in the previous theorem.*

(ii) The Cauchy problem (1.5.23) *is globally wellposed in* L^2

Proof. Identical to that of Theorem 1.4.12. □

As for the quintic resonant equation, Theorem 1.3.10 may be used to produce stationary solutions of the equation $iu_t = \mathcal{T}_4(u,u,u)$. In fact, if ϕ_n is a Hermite function we have, recalling that $G(x) = c_0\phi_0$,

$$T_{S(\theta)}(G, \phi_n, \phi_n, G, \phi_n) = c_0^2 T_{S(\theta)}(\phi_0, \phi_n, \phi_n, \phi_0, \phi_n) = C\phi_n,$$

for some constant C, by Theorem 1.3.10. This immediately implies that,

$$\mathcal{T}_4(\phi_n, \phi_n, \phi_n) = \frac{1}{2\sqrt{2}\pi^2}\int_0^{2\pi} T_{S(\theta)}(G, \phi_n, \phi_n, G, \phi_n)d\theta = C\phi_n,$$

and hence that ϕ_n is a stationary wave for the Hamiltonian flow of $\mathcal{H}_4$. By taking the inner product with respect to ϕ_n and using $\mathcal{H}_4(\phi_n) = \|e^{itH}\phi_n\|_{L_t^4 L_x^4} = \|\phi_n\|_{L^4}$, one finds that $C = \|\phi_n\|^4/4$.

By applying the symmetries of $\mathcal{H}_4$, we find that all functions of the form,

$$ae^{ibx}\phi_n(x+c), \tag{1.5.24}$$

are stationary waves for $a \in \mathbb{C}$ and $b, c \in \mathbb{R}$. The set of stationary waves we can construct for the cubic case is smaller than the set we can construct for the quintic case in (1.4.37), because the cubic equation has fewer symmetries.

1.5.4.1 · Regularity of stationary waves: technical issues

All of the stationary waves constructed in the previous subsection are analytic and exponentially decaying in space. In the remainder of this section we prove that all stationary waves that are in L^2 are automatically analytic and decay in space like $e^{-\alpha x^2}$ for some $\alpha > 0$. This is analogous to Corollary 1.4.16 for the quintic resonant equation. Recall that the proof of that result relied on two ingredients: a refined multilinear Strichartz estimate (1.4.39) and a weight transfer property (1.4.38).

The weight transfer property was a direct result of the weight transfer lemma, Lemma 1.3.13, for the functionals E_A. In the present case we encounter a problem when trying to replicate this: when we try to transfer weight in the functional $\mathcal{E}_4$ using Lemma 1.3.13, the weight also hits the Gaussians,

$$E_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4G_{\mu,\epsilon}) \le E_{B(\lambda)}(GG_{\mu,\epsilon}, f_1G_{\mu,\epsilon}, f_2G_{\mu,\epsilon}, GG_{\mu,\epsilon}, f_3G_{\mu,\epsilon}, f_4), \tag{1.5.25}$$

and the right hand side here can't be related back to $\mathcal{E}_4$. To get around this, we observe that,

$$G(x)G_{\mu,\epsilon}(x) = e^{-\frac{1}{2}x^2}e^{\mu x^2/(1+\epsilon x^2)} \le e^{(-\frac{1}{2}+\mu)x^2}, \tag{1.5.26}$$

which, if $\mu < 1/2$, is still decaying exponentially fast, and should be possible to handle in estimates.

Because of this consideration, we are led to define,

$$\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4) = \frac{1}{\sqrt{2}\pi^2}\int_{\mathbb{R}} \frac{1}{1+\lambda^2} E_{B(\lambda)}\left(e^{(-\frac{1}{2}+\mu)x^2}, f_1, f_2, e^{(-\frac{1}{2}+\mu)x^2}, f_3, f_4\right) d\lambda, \tag{1.5.27}$$

and we note that $\mathcal{E}_4^0 = \mathcal{E}_4$. We now proceed to develop the two ingredients for the stationary wave result, noting that both ingredients need to be developed for $\mathcal{E}_4^\mu$ and not just $\mathcal{E}_4$.

1.5.4.2 · *The weight transfer property*

Lemma 1.5.10. *(i) If $\mu < \frac{1}{2}$ and functions f_k are positive then there holds,*

$$\mathcal{E}_4\left(f_1, f_2, f_3, f_4 G_{\mu,\epsilon}\right) \leq \mathcal{E}^{\mu}\left(f_1 G_{\mu,\epsilon}(x), f_2 G_{\mu,\epsilon}(x), f_3 G_{\mu,\epsilon}(x), f_4\right). \tag{1.5.28}$$

(ii) If $\mu < \frac{1}{2}$ then there holds the bound,

$$|\mathcal{E}_4^{\mu}(f_1, f_2, f_3, f_4)| \leq \sqrt{\frac{\pi}{8}} \frac{1}{\sqrt{1-2\mu}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \tag{1.5.29}$$

Proof. (i) This is immediate from the computations in (1.5.25) and (1.5.26).

(ii) Boundedness is proved in the usual way,

$$\begin{aligned} |\mathcal{E}_4^{\mu}(f_1, f_2, f_3, f_4)| &\leq \frac{1}{\sqrt{8}\pi} \int_{\mathbb{R}} \frac{1}{1+\lambda^2} \left\| e^{(-\frac{1}{2}+\mu)x^2} \right\|_{L^2}^2 \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2} d\lambda \\ &\leq \frac{1}{\sqrt{8}\pi} \left(\int_{\mathbb{R}} \frac{1}{1+\lambda^2} d\lambda \right) \left(\int_{\mathbb{R}} e^{-(1-2\mu)x^2} dx \right) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \end{aligned}$$

Evaluating the integrals appearing here yields the result. □

1.5.4.3 · *Refined multilinear estimates*

As for the quintic resonant equation, the refined multilinear estimates we need can be determined in an elementary way using the representations (1.5.7) and (1.5.9) for $\mathcal{E}_4$.

Lemma 1.5.11. *There is an absolute constant C such that if f_1 and f_3 are supported in $B(0, R)^C$ and f_2 and f_4 are supported in $B(0, r)$, with $R > 4r$, then,*

$$|\mathcal{E}_4(f_1, f_2, f_3, f_4)| \leq \frac{C}{\sqrt{R}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \tag{1.5.30}$$

Proof. From (1.5.7) we have,

$$\begin{aligned} \mathcal{E}_4(f_1, f_2, f_3, f_4) = \frac{1}{2\pi^2} \int_{\mathbb{R}^4} & e^{-\frac{1}{2}[(\lambda v_2)^2 + v_1^2]} f_1(\lambda v_1 + v_3) f_2(v_2 + v_3) \\ & \overline{f_3(\lambda v_1 + v_2 + v_3) f_4(v_3)} dv_1 dv_2 dv_3 d\lambda. \end{aligned} \tag{1.5.31}$$

If the integrand here is non-zero, we necessarily have $|v_3| \leq r$, $|v_2 + v_3| \leq r$, and $|\lambda v_1 + v_3| \geq R$. This gives,

$$|\lambda v_1| \geq |\lambda v_1 + v_3| - |v_3| \geq \frac{R}{2} \quad \text{and} \quad |v_2| \leq |v_2 + v_3| + |v_3| \leq r. \tag{1.5.32}$$

We will use these inequalities to impose constraints on $|\lambda v_2 + v_1|$ and $|\lambda v_2 - v_1|$, which are the inputs to the Gaussins in representation (1.5.9). If we can ensure that these are large, the fast decay of the Gaussians will imply that $\mathcal{E}_4$ is small. By inspection, we see that large values of $|\lambda|$ pose a problem, but such large values can be dealt with separately by using the decay of $1/(1+\lambda^2)$ in (1.5.9).

Regime One: $|\lambda| \leq \sqrt{R}/4$. Observe that,

$$|\lambda v_2 + v_1| \geq |v_1| - |\lambda v_2| \geq \frac{R}{2|\lambda|} - 2|\lambda|,$$

in the last step using (1.5.32). Because the function $x \mapsto R/(2x) - 2x$ is decreasing for positive x, we have,

$$|\lambda v_2 + v_1| \geq \frac{R}{2|\lambda|} - 2|\lambda| \geq \frac{R}{2(\sqrt{R}/4)} - 2(\sqrt{R}/4) > \sqrt{R}.$$

An identical argument shows that $|\lambda v_2 - v_1| > \sqrt{R}$. It follows that,

$$\begin{aligned} A := & \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \leq \sqrt{R}/4} \frac{1}{1+\lambda^2} |E_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4)| d\lambda \\ = & \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \leq \sqrt{R}/4} \frac{1}{1+\lambda^2} |E_{B(\lambda)}(G\chi_{|x|\geq\sqrt{R}}, f_1, f_2, G\chi_{|x|\geq\sqrt{R}}, f_3, f_4)| d\lambda \\ \leq & \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} \frac{1}{1+\lambda^2} d\lambda \right) \|G\chi_{|x|\geq\sqrt{R}}\|_{L^2}^2 \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \end{aligned}$$

We estimate,

$$\|G\chi_{|x|\geq\sqrt{R}}\|_{L^2}^2 = 2\int_{\sqrt{R}}^{\infty} e^{-x^2} dx \leq \frac{2}{\sqrt{R}} \int_{\sqrt{R}}^{\infty} x e^{-x^2} dx = \frac{2}{\sqrt{R}} e^{-R} \leq \frac{2}{\sqrt{R}},$$

and hence,

$$A \leq \frac{C}{\sqrt{R}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2},$$

for some absolute constant C.

Regime Two: $|\lambda| \geq \sqrt{R}/4$. This is easier: we have,

$$\begin{aligned} B := & \frac{1}{\sqrt{2\pi}} \int_{|\lambda| \geq \sqrt{R}/4} \frac{1}{1+\lambda^2} |\mathcal{E}_{B(\lambda)}(G, f_1, f_2, G, f_3, f_4)| d\lambda \\ \leq & \frac{\sqrt{2}}{\pi} \left(\int_{\sqrt{R}/4}^{\infty} \frac{1}{1+\lambda^2} d\lambda \right) \|G\|_{L^2}^2 \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \end{aligned}$$

We estimate

$$\int_{\sqrt{R}/4}^{\infty} \frac{1}{1+\lambda^2} d\lambda \leq \int_{\sqrt{R}/4}^{\infty} \frac{1}{\lambda^2} d\lambda \leq= \frac{4}{\sqrt{R}},$$

which then gives,

$$B \leq \frac{C}{\sqrt{R}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.$$

Then, because $\mathcal{E}_4(f_1, f_2, f_3, f_4) = A + B$, equation (1.5.30) is established. □

Theorem 1.5.12. *There is an absolute constant C such that if $\mu \in [0, 1/2)$, f_k is supported in $B(0, r)$ and f_j is supported in $B(0, R)^C$, with $R > 4r$, then*

$$|\mathcal{E}_4^{\mu}(f_1, f_2, f_3, f_4)| \leq \frac{C}{(1-2\mu)^{5/8} R^{1/4}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}. \tag{1.5.33}$$

Proof. For fixed $\mu \in [0, 1/2)$, let $\alpha = \sqrt{(1/2) - \mu}$. We will again adopt the notation $f^{\lambda}(x) = \lambda^{1/2} f(\lambda x)$. With this notation we have,

$$e^{-(\frac{1}{2}-\mu)x^2} = e^{-(\alpha x)^2} = G(\alpha x) = \alpha^{-1/2} G^\alpha(x).$$

Using the scaling property of $E_{B(\lambda)}$, we have,

$$\begin{aligned}
\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4) &= \frac{1}{\sqrt{2}\pi} \int_{\mathbb{R}} \frac{1}{1+\lambda^2} E_{B(\lambda)}\left(\alpha^{-1/2} G^\alpha, f_1, f_2, \alpha^{-1/2} G^\alpha, f_3, f_4\right) \\
&= \frac{1}{\alpha} \frac{1}{\sqrt{2}\pi} \int_{\mathbb{R}} \frac{1}{1+\lambda^2} E_{B(\lambda)}\left(G, f_1^{1/\alpha}, f_2^{1/\alpha}, G, f_3^{1/\alpha}, f_4^{1/\alpha}\right) \\
&= \frac{1}{\alpha} \mathcal{E}_4\left(f_1^{1/\alpha}, f_2^{1/\alpha}, f_3^{1/\alpha}, f_4^{1/\alpha}\right).
\end{aligned}$$

Now assume that f_{k_1} is supported in $B(0,R)^C$ and f_{k_2} is supported in $B(0,r)$. We then have that $f_{k_1}^{1/\alpha}$ is supported in $B(0,\alpha R)^C$ and $f_{k_2}^{1/\alpha}$ is supported in $B(0,\alpha r)$. Then, using representation (1.5.2), we have,

$$\begin{aligned}
&|\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4)| \\
&\quad = \left|\frac{1}{\alpha} \mathcal{E}_4(f_1^{1/\alpha}, f_2^{1/\alpha}, f_3^{1/\alpha}, f_4^{1/\alpha})\right| \\
&\quad = \left|\frac{1}{\alpha} \frac{2}{\pi} \int_0^{\pi/2} \int_{\mathbb{R}} (e^{itH} f_1^{1/\alpha})(e^{itH} f_2^{1/\alpha})\overline{(e^{itH} f_3^{1/\alpha})(e^{itH} f_4^{1/\alpha})} dx dt\right| \\
&\quad \leq \frac{1}{\alpha}\left(\frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}} |(e^{itH} f_{k_1}^{1/\alpha})(e^{itH} f_{k_2}^{1/\alpha})|^2\right)^{1/2}\left(\frac{2}{\pi}\int_0^{\pi/2}\int_{\mathbb{R}} |(e^{itH} f_{k_3}^{1/\alpha})(e^{itH} f_{k_4}^{1/\alpha})|^2\right)^{1/2} \\
&\quad = \frac{1}{\alpha} \mathcal{E}_4(f_{k_1}^{1/\alpha}, f_{k_2}^{1/\alpha}, f_{k_1}^{1/\alpha}, f_{k_2}^{1/\alpha})^{1/2} \mathcal{E}_4(f_{k_3}^{1/\alpha}, f_{k_4}^{1/\alpha}, f_{k_3}^{1/\alpha}, f_{k_4}^{1/\alpha})^{1/2}
\end{aligned}$$

Using (1.5.30), we get,

$$\mathcal{E}_4(f_{k_1}^{1/\alpha}, f_{k_2}^{1/\alpha}, f_{k_1}^{1/\alpha}, f_{k_2}^{1/\alpha})^{1/2} \leq \frac{C}{(\alpha R)^{1/2}} \|f_{k_1}^{1/\alpha}\|_{L^2}^2 \|f_{k_2}^{1/\alpha}\|_{L^2}^2 = \frac{C}{(\alpha R)^{1/2}} \|f_{k_1}\|_{L^2}^2 \|f_{k_2}\|_{L^2}^2,$$

while for the other $\mathcal{E}_4$ term we can use the usual L^2 boundedness. This gives,

$$|\mathcal{E}_4^\mu(f_1, f_2, f_3, f_4)| \leq \frac{C}{\alpha^{5/4} R^{1/4}} \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2} \|f_4\|_{L^2}.$$

Substituting back in $\alpha = \sqrt{(1/2)-\mu}$ gives the result. □

1.5.4.4 · Stationary waves are analytic

Theorem 1.5.13. *Suppose that $\phi \in L^2$ is a stationary wave solution of $iu_t = \mathcal{T}_4(u,u,u)$; that is, ϕ satisfies,*

$$\omega\phi(x) = \mathcal{T}_4(\phi,\phi,\phi)(x), \tag{1.5.34}$$

for some ω. Then there exists $\alpha > 0$ and $\beta > 0$ such that $\phi e^{\alpha x^2} \in L^\infty$ and $\widehat{\phi} e^{\beta x^2} \in L^\infty$. As a result, ϕ can be extended to an entire function on the complex plane.

Using the proof of Corollary 1.4.16, this theorem is an immediate consequence of the following proposition.

Proposition 1.5.14. *Suppose that $\phi \in L^2$ satisfies,*

$$\omega|\phi(x)| \leq \mathcal{T}_4(|\phi|, |\phi|, |\phi|)(x), \tag{1.5.35}$$

for some $\omega > 0$. Then there exists $\alpha > 0$ such that $x \mapsto \phi(x)e^{\alpha x^2} \in L^2$.

Proof of proposition. For the proof, we will find μ so that we have the bound $\|\phi G_{\mu,\epsilon}\|_{L^2} \lesssim 1$ independently of ϵ. Taking the limit $\epsilon \to 0$ will the yield the result. The structure of proof here is extremely similar to that of Theorem 1.4.15. For brevity, we will only describe the start of the proof here, which is the only part that is essentially different to the proof of Theorem 1.4.15.

First, we fix throughout $\mu \leq 1/4$. Using formulas (1.5.28), (1.5.29) and (1.5.33), there are constants C independent of μ, such that,

$$\mathcal{E}_4(f_1, f_2, f_3, f_4 G_{\mu,\epsilon}) \leq \mathcal{E}_4^{\mu}(f_1 G_{\mu,\epsilon}, f_2 G_{\mu,\epsilon}, f_3 G_{\mu,\epsilon}, f_4) \tag{1.5.36}$$

$$|\mathcal{E}_4^{\mu}(f_1, f_2, f_3, f_4)| \leq C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2}\|f_4\|_{L^2} \tag{1.5.37}$$

$$|\mathcal{E}_4^{\mu}(f_1, f_2, f_3, f_4)| \leq C\frac{1}{R^{1/4}}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2}\|f_4\|_{L^2}, \tag{1.5.38}$$

where in the last inequality, f_i is supported in $B(0,r)$ and f_j is supported in $B(0,R)^C$ for $R > 4r$.

Now consider a function ϕ satisfying (1.5.35). We may assume ϕ is non-negative. For any $M > 0$ define,

$$\phi_<(x) = \phi(x)\chi_{|x|\leq M}(x), \qquad \phi_\sim(x) = \phi(x)\chi_{M<|x|\leq M^2}(x), \qquad \phi_>(x) = \phi(x)\chi_{|x|\leq M^2}(x).$$

We have the decomposition $\phi = \phi_> + \phi_\sim + \phi_>$, and the supports are all disjoint, which gives

$$\|\phi G_{\mu,\epsilon}\|_{L^2}^2 = \|\phi_< G_{\mu,\epsilon}\|_{L^2}^2 + \|\phi_\sim G_{\mu,\epsilon}\|_{L^2}^2 + \|\phi_> G_{\mu,\epsilon}\|_{L^2}^2.$$

The first two terms are trivial to to bound uniformely in M. If $|x| \leq M^2$, we have,

$$G_{\mu,\epsilon}(x) \leq e^{\mu x^2} \leq e^{\mu M^4}$$

so setting $\mu = M^{-4}$ gives, $\|\phi_< G_{\mu,\epsilon}\|_{L^2} \leq \|\phi_< e^1\|_{L^2} \leq e^1\|\phi\|_{L^2} \lesssim 1$, with the same bound for $\phi_\sim$. In order to prove the theorem, it remains then to bound $\|\phi_> e^{G_{\mu,\epsilon,n}}\|_{L^2}$.

Starting with the equation (1.5.35) of the theorem, we multiply both sides by $\phi_>(x)G_{\mu,\epsilon}(x)^2$ which gives,

$$\omega\phi_>(x)^2 G_{\mu,\epsilon}(x)^2 \leq \mathcal{T}_4(\phi, \dots, \phi)(x)\phi_>(x)G_{\mu,\epsilon}(x)^2.$$

Now integrating over $\mathbb{R}$ and using (1.5.36) gives,

$$\omega\|\phi_> G_{\mu,\epsilon}\|_{L^2}^2 \leq \mathcal{E}_4(\phi, \phi, \phi, \phi_> G_{\mu,\epsilon}{}^2) \leq \mathcal{E}_4^{\mu}(\phi G_{\mu,\epsilon}, \phi G_{\mu,\epsilon}, \phi G_{\mu,\epsilon}, \phi_> G_{\mu,\epsilon}).$$

For convenience, let $\psi = \phi G_{\mu,\epsilon}$. The bound then reads,

$$\omega\|\psi_>\|_{L^2}^2 \lesssim \mathcal{E}_4^{\mu}(\psi, \psi, \psi, \psi_>).$$

Now write each $\psi = \psi_< + \psi_\sim + \psi_>$ and expand the multinear functional. We will get many terms, which we bound in one of two ways.

- If there are three or more $\psi_>$ terms, bound by $\|\psi_>\|_{L^2}^k$ where k is the number of $\psi_>$ terms appearing, using (1.5.37). In this case the other terms are $\psi_<$ or $\psi_\sim$, which are uniformly bounded.
- If there are one or two $\psi_>$ terms, then there is either a $\psi_<$ term or a $\psi_\sim$ term. In the former case we can use the refined multilinear estimate (1.5.38), with $R = M^2$, and bound by $M^{-1/2}\|\psi_>\|^k$ (where $k = 1$ or $k = 2$). In the latter case we can bound by $\|\psi_\sim\|_{L^2}\|\psi_>\|_{L^2}^k \lesssim \|\phi_\sim\|_{L^2}\|\psi_>\|_{L^2}^k$ using (1.5.36).

In total, we get,

$$\begin{aligned}\omega\|\psi_>\|_{L^2}^2 &\leq \mathcal{E}_4^\mu(\psi,\ldots,\psi,\psi_>)\\ &\leq C\left(\|\psi_>\|_{L^2}^4 + \|\psi_>\|_{L^2}^3 + (M^{-1/2} + \|\phi_\sim\|_{L^2})(\|\psi_>\|_{L^2}^2 + \|\psi_>\|_{L^2})\right),\end{aligned}$$

for a constant C independent of μ. This formula has the same structure as equation (1.4.42) in the proof of Theorem 1.4.15. Replicating the same argument there, we find that if we choose M sufficiently large there is a constant independent of ϵ such that $\|\psi_>\|_{L^2} \leq C$. Letting $\epsilon \to 0$ then gives the result. □

PROOF OF THEOREM 1.3.12

Theorem 1.3.12 has two distinct parts, which we state here independently as Lemma 1.5.15 and Theorem 1.5.16.

Lemma 1.5.15 (Decomposition Lemma)**.** *Let A be an isometry. Denote $\tilde{f}_k(x) = f_k(-x)$. There exists integers m and l, with $0 \leq m \leq l \leq n$, and two permutations σ_1 and σ_2 of the integers $\{1,\ldots,n\}$ such that,*

$$\begin{aligned}E_A(f_1,\ldots,f_{2n}) = &\left(\prod_{k=1}^{m}\langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)}\rangle\right)\left(\prod_{k=m+1}^{l}\langle f_{\sigma_1(k)}, \tilde{f}_{n+\sigma_2(k)}\rangle\right)\\ &\times E_B(f_{\sigma_1(l+1)},\ldots f_{\sigma_1(n)}, f_{n+\sigma_2(l+1)},\ldots f_{n+\sigma_2(n)}),\end{aligned}$$

where the matrix $B : \mathbb{R}^{n-l} \to \mathbb{R}^{n-l}$ has no permutation part; that is, for all i and j, $Be_i \neq \pm e_j$.

Proof. Call a pair of integers (i,j) good if $Ae_i = e_j$ and bad if $Ae_i = -e_j$. Because A is injective, for a given j there is at most one i such that (i,j) is good or bad. Let m be the number of good pairs, and $l-m$ the number of bad pairs. Order the good pairs in any way, and for $k = 1,\ldots,m$, let $\sigma_1(k) = i$ and $\sigma_2(k) = j$ where (i,j) is the kth good pair in the ordering. In a similar fashion, order the bad pairs in any way, and for $k = l+1,\ldots,m$, let $\sigma_1(k) = i$ and $\sigma_2(k) = j$ where (i,j) is the $(k-m)$th bad pair in the ordering.

Now consider i such that i is not the first component in a good or bad pair. There are $n-l$ such i. Order them in any way, and for $k = l+1,\ldots,n$ set $\sigma_1(k) = i$ where i is the $(k-l)$th number in the ordering. Then consider j such that j is not the second component in a good or bad pair, and for $k = l+1,\ldots,n$ define $\sigma_2(k) = j$ in a similar fashion.

It is clear that,

$$A : \text{span}(e_{\sigma_1(1)},\ldots,e_{\sigma_1(m)}) \cong \mathbb{R}^m \to \text{span}(e_{\sigma_2(1)},\ldots,e_{\sigma_2(m)}) \cong \mathbb{R}^m \tag{1.5.39}$$

$$A : \text{span}(e_{\sigma_1(m+1)},\ldots,e_{\sigma_1(l)}) \cong \mathbb{R}^{l-m} \to \text{span}(e_{\sigma_2(m+1)},\ldots,e_{\sigma_2(l)}) \cong \mathbb{R}^{l-m}, \tag{1.5.40}$$

and that $A = I$ and $A = -I$ on these subspaces respectively, with the implied identifications of bases. Because A is an isometry, we then necessarily have,

$$A : \operatorname{span}(e_{\sigma_1(m+1)}, \dots, e_{\sigma_1(n)}) \cong \mathbb{R}^{n-m} \to \operatorname{span}(e_{\sigma_2(m+1)}, \dots, e_{\sigma_2(n)}) \cong \mathbb{R}^{n-m}. \tag{1.5.41}$$

Let B be the restriction of A as a map between the subspaces in (1.5.41). The condition that $Be_i \neq \pm e_j$ holds because otherwise (i, j) would be a good or bad pair and e_i and e_j would not be in the subspaces given in (1.5.41).

The representation of E_A arises because A is the identity in (1.5.39) and the negative of the identity in (1.5.40). □

In light of the representation of E_A, it is clear that we have the equality $|E_A(f_1, \dots, f_{2n})| = \prod_{k=1}^{2n} \|f_k\|_{L^2}$ if and only if the following three conditions hold.

1. For all $k = 1, \dots, m$, $|\langle f_{\sigma_1(k)}, f_{n+\sigma_2(k)}\rangle| = \|f_{\sigma_1(k)}\|_{L^2} \|f_{n+\sigma_2(k)}\|_{L^2}$, which means $f_{\sigma_1(k)} = C_k f_{n+\sigma_2(k)}$ for some constant C_k by the usual Cauchy–Schwarz equality condition.

2. For all $k = m+1, \dots, l$,

$$|\langle f_{\sigma_1(k)}, \tilde{f}_{n+\sigma_2(k)}\rangle| = \|f_{\sigma_1(k)}\|_{L^2} \|\tilde{f}_{n+\sigma_2(k)}\|_{L^2} = \|f_{\sigma_1(k)}\|_{L^2} \|f_{n+\sigma_2(k)}\|_{L^2},$$

   which means $f_{\sigma_1(k)} = C_k \tilde{f}_{n+\sigma_2(k)}$ for some constant C_k, again by the usual Cauchy–Schwarz equality condition.

3. $|E_B(f_{\sigma_1(l+1)}, \dots f_{\sigma_1(n)}, f_{n+\sigma_2(l+1)}, \dots f_{n+\sigma_2(n)})| = \prod_{k=l+1}^{n} \|f_{\sigma_1(k)}\|_{L^2} \|f_{n+\sigma_2(k)}\|_{L^2}$

To finish the proof of Theorem 1.3.12, we examine the equality case in item 3. This equality is, of course, identical looking to the original equality condition (1.3.20). The difference is that B has the structural condition $Be_i \neq \pm e_j$.

Theorem 1.5.16. *Suppose that A is an isometry, and for all i and j, $Ae_i \neq \pm e_j$. Then,*

$$|E_A(f_1, \dots, f_{2n})| = \prod_{k=1}^{2n} \|f_k\|_{L^2},$$

only if each of the functions is a Gaussian. Equality holds if each of the functions is the same Gaussian of the form $e^{-\alpha x^2}$ for some $\alpha > 0$.

The 'if' part of the Theorem was proved before, in Theorem 1.3.11. The 'only if' part follows from a close analysis of the Cauchy–Schwarz equality condition,

$$\prod_{k=1}^{n} f_k((Ax)_k) = \prod_{k=1}^{n} f_{n+k}(x_k), \tag{1.5.42}$$

which was given in (1.3.20). The proof is in three steps.

1. First we show that if all the functions in (1.5.42) are smooth and strictly positive, then they must be Gaussians.

2. Next we show that if all the functions in (1.5.42) are merely non-negative and in L^2, then they must be Gaussians. The idea here is extremely simple: the heat flow acting on each function simultaneously conserves the relationship (1.5.42). For future times, the heat flows are smooth and positive, and part one can be applied.

3. Finally we show that if the functions in (1.5.42) are complex valued and in L^2, then they must be Gaussians.

Lemma 1.5.17 (Step One)**.** *Suppose that functions f_k satisfy* (1.5.42)*, are all strictly positive and smooth. Then they are all Gaussians.*

Proof. Let $g_k(x) = \log f_k(x)$. Because f_k is strictly positive and smooth, g_k is well-defined and smooth. To show f_k is a Gaussian we will show that g_k is polynomial of degree at most two.

Because A is an isometry, it satisfies $A^{-1} = A^T$. Therefore, for all m, we have the matrix expansions,

$$Ae_m = \sum_{k=1}^{n} a_{km}e_k \quad \text{and} \quad A^{-1}e_m = \sum_{k=1}^{n} a_{mk}e_k, \tag{1.5.43}$$

for numbers $\{a_{km}\}_{k,m=1}^{n}$. By the assumptions in the theorem, we have $|a_{km}| < 1$ for all k and all m. Set $\epsilon = \max_{m,k} |a_{km}| < 1$. Because A is an isometry, we have,

$$1 = |Ae_m|^2 = \sum_{k=1}^{n} a_{km}^2 \quad \text{and} \quad 1 = |A^{-1}e_m|^2 = \sum_{k=1}^{n} a_{mk}^2, \tag{1.5.44}$$

for all m.

Taking the logarithm of both sides of the Cauchy–Schwarz equality condition (1.5.42) (and switching the left and right sides) gives the condition on the functions g_k,

$$\sum_{k=1}^{n} g_{n+k}(x_k) = \sum_{k=1}^{n} g_k((Ax)_k). \tag{1.5.45}$$

Fix $m \in \{1, \dots, n\}$ and set $x = te_m$. For all $k \in \{1, \dots, n\}$, we have $(Ax)_k = t(Ae_m)_k = ta_{mk}$ and $x_k = t\delta_{km}$. Therefore,

$$g_{n+m}(t) + \sum_{\substack{k=1\\k\neq m}}^{n} g_{n+k}(0) = \sum_{k=1}^{n} g_k(a_{km}t).$$

Differentiating this equation twice with respect to t yields,

$$g''_{n+m}(t) = \sum_{k=1}^{n} a_{km}^2 g''_k(a_{km}t). \tag{1.5.46}$$

We can evaluate this equation at $t = 0$ to get $g''_{n+m}(0) = \sum_{k=1}^{n} a_{km}^2 g''_k(0)$. Subtracting this from (1.5.46) then gives,

$$g''_{n+m}(t) - g''_{n+m}(0) = \sum_{k=1}^{n} a_{km}^2 \left[g''_k(a_{km}t) - g''_k(0)\right]. \tag{1.5.47}$$

For all indices m and k we have $a_{km}t \in [-\epsilon|t|, \epsilon|t|]$ and so the bound,

$$|g''_{n+m}(t) - g''_{n+m}(0)| \le \max_{s\in[-\epsilon|t|,\epsilon|t|]} \max_{k=1,\dots,n} |g''_k(s) - g''_k(0)|, \tag{1.5.48}$$

holds for all $m \in \{1, \dots, n\}$.

The Cauchy–Schwarz equality condition is similar when the roles of $f_1, \dots, f_n$ and $f_{n+1}, \dots, f_{2n}$ are switched.

This may be seen by replacing x by Ax and finding,

$$\prod_{k=1}^{n} f_{n+k}((A^{-1}x)_k) = \prod_{k=1}^{n} f_k(x_k), \tag{1.5.49}$$

By equations (1.5.43) and (1.5.44), the same estimates for A hold for A^{-1}. Therefore, performing the same argument as before, we find,

$$|g_m''(t) - g_m''(0)| \le \max_{s\in[-\epsilon|t|,\epsilon|t|]} \max_{k=1,\dots,n} |g_{n+k}''(s) - g_{n+k}''(0)| \tag{1.5.50}$$

for all $m \in \{1, \dots, n\}$.

The two families of inequalities (1.5.48) and (1.5.50) may be combined into one inequality,

$$\max_{k=1,\dots,2n} |g_k''(t) - g_k''(0)| \le \max_{s\in[-\epsilon|t|,\epsilon|t|]} \max_{k=1,\dots,2n} |g_k''(s) - g_k''(0)| \tag{1.5.51}$$

This holds for all t. Applying it recursively N times yields,

$$\max_{k=1,\dots,2n} |g_k''(t) - g_k''(0)| \le \max_{s\in[-\epsilon^N|t|,\epsilon^N|t|]} \max_{k=1,\dots,2n} |g_k''(s) - g_k''(0)|.$$

But now $\epsilon < 1$, and so we have, because g is smooth,

$$\begin{aligned}\max_{k=1,\dots,2n} |g_k''(t) - g_k''(0)| &\le \lim_{N\to\infty} \max_{s\in[-\epsilon^N|t|,\epsilon^N|t|]} \max_{k=1,\dots,2n} |g_k''(s) - g_k''(0)| \\ &= \max_{k=1,\dots,2n} |g_k''(0) - g_k''(0)| = 0.\end{aligned}$$

Hence $g_k''(x) = g_k''(0)$ is a constant, $g_k(x)$ is a polynomial of degree at most two, and $f_k(x) = e^{g_k(x)}$ is a Gaussian. □

The proof of the previous lemma required smoothness and positivity assumptions on the f_k functions. We now use a heat flow argument to upgrade the result to non-smooth and non-negative f_k.

Lemma 1.5.18 (Step two, part one)**.** *The Cauchy–Schwarz equality condition* (1.5.42) *is conserved by the heat flow. More precisely, suppose that there is a $\gamma > 0$ such that all of the functions in* (1.5.42) *satisfy $f_k e^{-\gamma x^2} \in L^1$. Then there is a time T such that the heat flow $e^{-t\Delta} f_k$ is defined for $t \in (0, T)$ and all k. For fixed $t \in (0, T)$ the functions $h_k(x) = (e^{-t\Delta} f_k)(x)$ satisfy* (1.5.42).

Proof. Under the assumptions on f_k we can write the solution of the heat equation $e^{-t\Delta} f_k$ using the fundemental solution as,

$$(e^{-t\Delta} f_k)(x_k) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-|y_1|^2/4t} f(x_k - y_k) dy_k.$$

The formula is well defined for $0 < t < T$, with $T < 1/\gamma$, because of the integrability assumptions on f_k. Using this formula n times, we have,

$$\begin{aligned}\prod_{k=1}^{n}(e^{-t\Delta}f_k)((Ax)_k) &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/4t}\prod_{k=1}^{n} f_k((Ax)_k - y_k)dy \\ &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/4t}\prod_{k=1}^{n} f_k((A(x-A^{-1}y))_k)dy \\ &= \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2/4t}\prod_{k=1}^{n} f_{n+k}((x-A^{-1}y)_k)dy,\end{aligned}$$

where in the last line we have used the Cauchy–Schwarz equality condition (1.5.42). We now perform the change of variables $z = A^{-1}y$. Because A is an isometry, the determinint of this change of variables is 1, and we also have $|y| = |Az| = |z|$ for every $y \in \mathbb{R}^n$. This gives,

$$\prod_{k=1}^{n}(e^{-t\Delta}f_k)((Ax)_k) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/4t}\prod_{k=1}^{n} f_{n+k}(x_k - z_k)dy = \prod_{k=1}^{n}(e^{-t\Delta}f_{n+k})(x_k).$$

For fixed t the maps $h_k(x) = e^{-t\Delta}f(x)$ thus satisfy (1.5.42). □

Corollary 1.5.19 (Part two, step two)**.** *Suppose that non-negative functions $f_k \in L^2$ satisfy* (1.5.42)*. Then all of the functions are Gaussians.*

Proof. If $f_k \in L^2$, then $f_k e^{-\gamma x^2} \in L^1$ for all $\gamma > 0$. By the previous lemma, the heat flow $e^{-t\Delta}f_k$ exists for all k and $t \in (0,\infty)$, and the functions $h_k(x) = e^{-t\Delta}f_k(x)$ satisfy (1.5.42). The functions h_k are smooth, and because the initial data is non-negative, the functions h_k are also positive. By the first lemma, each of the h_k functions are Gaussians.

Fix k. By substituting the general form of a time dependent Gaussian into the heat equation we discover that if $e^{t\Delta}f_k$ is a Gaussian for all $t \in (0,T)$ then necessarily, for $t > 0$,

$$(e^{-t\Delta}f_k)(x) = \frac{d}{\sqrt{t+a}}e^{-b(x-c)^2/4(t+a)},$$

for some $a \geq 0$ and $b, d > 0$ and $c \in \mathbb{R}$. We calculate,

$$\|e^{-t\Delta}f_k\|_{L^2}^2 = \frac{d^2}{t+a}\int_{\mathbb{R}} e^{-b(x-c)^2/2(t+a)}dx = \frac{d^2}{\sqrt{t+a}}\sqrt{\frac{2\pi}{b}}.$$

Then, because $\|e^{-t\Delta}f_k\|_{L^2} \leq \|f_k\|_{L^2}$, we must have $a > 0$. This gives,

$$f_k(x) = \lim_{t\to 0}(e^{-t\Delta}f_k)(x) = \frac{1}{\sqrt{a}}e^{(bx^2+cx)/4a},$$

so f_k is a Gaussian. □

Having proved the result for non-negative f_k we lastly prove it for general complex valued f_k.

Lemma 1.5.20 (Step three)**.** *Suppose that functions $f_k \in L^2$ satisfy* (1.5.42)*. Then all of the functions are Gaussians.*

Proof. We assume that the functions f_k are smooth; the result for general functions follows from invoking the heat flow argument as in Step Two.

Taking absolute values in the equality condition (1.5.42), we see that the functions $|f_k|$ satisfy the condition as soon as the functions f_k do. By the previous Lemmas, $|f_k|$ must be a Gaussian $G_k(x)$ for all k, and hence,

$$f_k(x) = e^{i g_k(x)} G_k(x), \tag{1.5.52}$$

for some real valued function g_k. Note that from this we necessarily have $f_k(x) \neq 0$. Because f_k and G_k are smooth, and $e^{i g_k(x)} = f_k(x)/G_k(x)$ we can choose g_k to be smooth.

Plugging the expressions (1.5.52) for f_k into (1.5.42) we find,

$$\prod_{k=1}^{n} e^{i g_k((Ax)_k)} G_k((Ax)_k) = \prod_{k=1}^{n} e^{i g_{n+k}(x_k)} G_{n+k}(x_k).$$

The G_k terms cancel because we know the functions $G_k = |f_k|$ also satisfy (1.5.42). We are left with $e^{i \sum_{k=1}^{n} g_k((Ax)_k)} = e^{i \sum_{k=1}^{n} g_k(x_k)}$, and hence,

$$\sum_{k=1}^{n} g_k((Ax)_k) = \sum_{k=1}^{n} g_{n+k}(x_k) + 2\pi n(x).$$

where $n : \mathbb{R}^n \to \mathbb{Z}$. Because g is smooth, n is smooth and hence a constant. Plugging in $x = 0$ gives $n(x) = 0$, and then the equation for the functions g_k is thus,

$$\sum_{k=1}^{n} g_k((Ax)_k) = \sum_{k=1}^{n} g_{n+k}(x_k).$$

This is precisely equation (1.5.45). As before, g_k must be a polynomial of degree at most two, and f_k is a complex Gaussian. $\square$

Chapter 2

Schrödinger Maps

§2.1 · INTRODUCTION

The harmonic map heat flow and the Schrödinger maps equations are natural generalizations of the linear heat and Schrödinger equations where the domain and range of the functions considered are manifolds and the Euclidean partial derivatives are replaced by covariant derivatives. In this article we will be exclusively discussing the setting when the base space is some Euclidean space $\mathbb{R}^d$ and the target is a Kähler manifold N with complex structure J. The energy of a map $u : \mathbb{R}^d \to N$ is defined by the formula,

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^d} |du|^2 dV.$$

The Euler Lagrange operator $\tau(u)$ corresponding to $\mathcal{E}$ is calculated, in coordinates, to be $\tau(u) = \sum_{k=1}^{d} D_k \partial_k u$, where the D_k operators are covariant derivatives on N. The *harmonic map heat flow* is then the Cauchy problem given by,

$$u_t = \tau(u) = \sum_k D_k \partial_k u, \quad u(0) = u_0, \tag{2.1.1}$$

while the *Schrödinger maps equation* is the Cauchy problem given by,

$$u_t = J\tau(u) = J\sum_k D_k \partial_k u, \quad u(0) = u_0. \tag{2.1.2}$$

One can also consider the *generalized Landau-Lifshitz (GLL) equation*, defined for $\alpha \in [0,\infty)$ and $\beta \in \mathbb{R}$ by,

$$u_t = J\tau(u) = (\alpha + \beta J)\sum_k D_k \partial_k u, \quad u(0) = u_0; \tag{2.1.3}$$

this corresponds, when the range is $\mathbb{C}$, to the PDE $(\alpha + \beta i)u_t = \Delta u$. Let us emphasize that the linearity of the equations in the familiar case when the target is $\mathbb{C}$ is special: in general these problems are nonlinear because of curvature.

The harmonic map heat flow is a well known and intensively studied problem. It was introduced in [17] as a tool for studying the existence of harmonic maps. These are maps which satisfy $\tau(u) = D_k \partial_k u = 0$ and correspond to stationary solutions of all of the problems above. Vast work done on the harmonic map heat flow in the subsequent years; see, for example, [37] for a textbook treatment. We mention only that it has been shown that for general N uniqueness of the harmonic map heat flow does not hold, and that one way to demonstrate non-uniqueness is through studying self-similar solutions, as is done in [26, 23]. This approach is used to prove a non-uniqueness result for the case of the flow for maps from $\mathbb{C}^2 \cong \mathbb{R}^4$ to $\mathbb{CP}^2$ in section 2.5.

As opposed to the harmonic map heat flow, the Schrödinger maps equation (2.1.2) has been much less studied in general. For the setting we are considering here, that of the flow for maps $u : \mathbb{R}^d \to N$, local well-posedness in the Sobolev space $H^l(\mathbb{R}^d; N)$ for integer $l > d/2 + 1$ is established in [38]. One can see by scaling that $\dot{H}^{d/2}$ is critical for the problem, and significant work has been done on proving global well-posedness in this and other critical spaces in the special case when the target is the sphere $N = \mathbb{S}^2$ [5, 6, 4, 34].

The case of the sphere is particularly attractive for two reasons. First, given the usual embedding $\mathbb{S}^2 \subset \mathbb{R}^3$, the Schrödinger maps equation becomes quite explicit. In this framework, the complex structure at the point u is simply given by the cross product in $\mathbb{R}^3$, $Jw = u \times w$. The derivative term is calculated to be $\sum_k D_k \partial_k u = \Delta u + |\nabla u|^2 u$, where Δ and ∇ are the Laplacian and gradient operators for functions from $\mathbb{R}^d$ to $\mathbb{R}^3$. The Schrödinger maps equation thus becomes,

$$u_t = u \times (\Delta u + |\nabla u|^2 u), \quad x \in \mathbb{R}^d, \quad u(x) \in \mathbb{S}^2 \subset \mathbb{R}^3. \tag{2.1.4}$$

The second reason this case of the Schrödinger maps equation is appealing is that it is physically relevant. Equation (2.1.4) is used to describe the dynamics of ferromagnetic spin systems, and is known in the physics community as the Heisenberg model. It is a special case of the equation,

$$u_t = (\alpha + \beta u \times)(\Delta u + |\nabla u|^2 u), \quad x \in \mathbb{R}^d, \tag{2.1.5}$$

which is the Landau-Lifshitz-Gilbert equation and is used to study the direction of magnetism in a solid. (The survey article [36] discusses the physical relevance of these equations.) The equation (2.1.5) corresponds precisely to the GLL equation (2.1.3) in the case of maps $u : \mathbb{R}^d \to \mathbb{S}^2$. The work on small data existence and uniqueness in a critical space for the Schrödinger maps equation in this case of the sphere culminated in [4], which furnished a global critical small data well-posedness result in the Sobolev space $\dot{H}^{d/2}$.

A large body of work has been devoted to the sphere problem when the domain is $\mathbb{R}^2$. The critical space is $\dot{H}^1$, so the problem in this dimension is energy critical. It is also tractable to study because one can make an *equivariant ansatz* and thereby study a sub-problem of the flow as a whole. The equivariant ansatz involves studying solutions of the form the form $u(r, \theta) = e^{m\theta R} f(r)$ where $f(r) \in \mathbb{R}^3$, $m \in \mathbb{Z}$, and R is the generator of rotations about the z-axis and given by the matrix,

$$R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The overall picture that has developed can be described in terms of the harmonic maps, which have finite energy in this context, and whose existence is generally seen as barrier to global well-posedness. In the case of radial maps, $m = 0$, there are no non-trivial harmonic maps and a global existence result for arbitrarily sized data in H^2 has been established [30]. In the case when $m = 1$, the lowest energy level of the non-trivial harmonic maps is 4π; for initial data with energy strictly smaller than this, global existence has been shown to hold [3]. On the other hand, in [39], a set of initial data with energy arbitrarily close to 4π is constructed which generates finite time blow up solutions. (This paper resolved the long standing question of whether finite energy initial data could lead to finite time blow up.) Finite time blow up solutions are also constructed in [40]. For $m \geq 3$, it has been shown that if the initial data has energy close to that of the harmonic maps then the solution is, in fact, global [30].

Still in dimension 2, the equivariant ansatz can be made under the more general assumption that the target N is a complex surface with an $\mathbb{S}^1$ symmetry. This was originally done in [14], where a critical well-posedness theory for equivariant data small in $\dot{H}^1$ was developed. Under the same equivariant ansatz, [27] take a different approach than the Sobolev theory, and instead study the self-similar solutions of the flow. These are solutions of the form $u(x,t) = \psi(x/\sqrt{|t|})$ for a profile ψ. A family of such solutions with C^∞ profiles is constructed, giving an example of regularity breakdown: these solutions are smooth at all times $t \neq 0$ but not smooth at $t = 0$. The study of these self similar solutions is supplemented with a global critical small data well-posedness theorem in a Lorentz space that is shown to include the self-similar data.

When the dimensions of the range and domain are larger than two, but the same, it is still possible to formulate an equivariant ansatz, as will be shown in detail below. For the case of the Schrödinger maps equation for maps $u : \mathbb{C}^n \to \mathbb{CP}^n$, this equivariant ansatz is considered in [16], where the existence of self-similar solutions is established.

The primary purpose of the present chapter is to expand upon this previous work on the equivariant $\mathbb{C}^n$ to $\mathbb{CP}^n$ case. We are particularly interested in establishing a critical global small data well-posedness result in a space. We are not constrained to the Schrödinger maps equation, and many of our results, including our analysis of the existence and dynamics of the self-similar solutions, are valid for the GLL equation. More broadly, our hope is to introduce and cast this problem in a transparent way in order to open it up to the kinds of investigations that have, thus far, been limited to the case of the sphere.

2.1.1 · Overview of the results

2.1.1.1 · The equivariant ansatz and derivation of the equation

We consider maps $v : \mathbb{C}^n \to \mathbb{CP}^n$, where $\mathbb{CP}^n$ is equipped with the Fubini-Study metric, for $n \geq 2$. The $n = 1$ case is the usual problem of $\mathbb{R}^2$ to the sphere because $\mathbb{CP}^1$ with the Fubini-Study metric is isometric to $\mathbb{S}^2$. In what follows n is the complex dimension and $d = 2n$ is the real dimension.

Recall that $\mathbb{CP}^n$ can be viewed in terms of the homogeneous coordinates as points $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ under the identification,

$$[z_0, z_1, \ldots, z_n] = [\alpha z_0, \alpha z_1, \ldots, \alpha z_n],$$

for all $\alpha \in \mathbb{C}\backslash\{0\}$. Given a complex isometry A of $\mathbb{C}^n$ we can construct an isometry $\tilde{A}$ of $\mathbb{CP}^n$ by the formula,

$$\tilde{A}[z_0, z_1, \ldots, z_n] = [z_0, A(z_1, \ldots, z_n)];$$

that is, we let A act on the last n coordinates in the homogeneous representation. A map $v : \mathbb{C}^n \to \mathbb{CP}^n$ is said to be *equivariant* if $v(Az) = \tilde{A}v(z)$ for all isometries A of $\mathbb{C}^n$ and all points $z \in \mathbb{C}^n$. This ansatz is formally conserved by the flow. This assumption is strong and, as we show, implies that v is in fact of the form,

$$v(z) = v((z_1, \ldots, z_n)) = [z_0, f(r)z_1, \ldots, f(r)z_n]$$

where $r = |z|$ and $f : \mathbb{R}^+ \to \mathbb{C}$. We observe that for any $x \in \mathbb{R}$ we have,

$$v((x, 0, \ldots, 0)) = [z_0, f(r)x, 0, \ldots, 0];$$

or namely that,

$$v(\mathbb{R}^+e_1) \subset \{[z_0, z_1, 0, \dots, 0] \;:\; z_0, z_1 \in \mathbb{C}\} \simeq \mathbb{CP}^1,$$

so the image of a real ray is contained in a complex line. The Fubini-Study metric of $\mathbb{CP}^n$ restricts to the Fubini-Study metric on this $\mathbb{CP}^1$, so in fact the image of $v(\mathbb{R}^+e_1)$ is contained in a manifold isometric to $\mathbb{S}^2$. The idea, now, is to parameterize this sphere in the usual embedding $\mathbb{S}^2 \subset \mathbb{R}^3$ and determine an equation on $u(r) = v(re_1) \in \mathbb{S}^2$. From the equivariant ansatz we can recover v from u.

By a computation we determine that the energy of v is given in terms of $u : \mathbb{C}^n \to \mathbb{S}^2$ by the formula,

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \left(|u_r|^2 + \frac{u_1^2 + u_2^2 + (2n-2)|u - e_3|^2}{r^2} \right) dx, \tag{2.1.6}$$

where $|u - e_3|$ is the Euclidean distance in $\mathbb{R}^3$ between u and the north pole of the sphere e_3, and $|u_r|$ is the Euclidean norm in $\mathbb{R}^3$ of u_r. Observe that in the case $n = 1$, we recover the usual energy for the equivariant $\mathbb{R}^2 \to \mathbb{S}^2$ problem, as we would expect. (See, for example, [3], p. 2.) For $n \geq 2$, one determines that any function u with finite energy is continuous and has a limit as $r \to \infty$; by inspecting the energy one sees that this limit must be the north pole e_3.

The harmonic map heat flow, the Schrödinger maps equation, and the GLL equation for this equivariant case are now determined by calculating the variation of the energy. We find that the GLL equation is given by,

$$u_t = (\alpha P + \beta u\times) \left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2} e_3 \right), \tag{2.1.7}$$

where P is the projection onto the tangent space $T_u\mathbb{S}^2$ and $u_3 = \langle u, e_3 \rangle$. The harmonic map heat flow corresponds to $\alpha = 1$ and $\beta = 0$; while the Schrödinger maps equation corresponds to $\alpha = 0$ and $\beta = 1$. This representation of the problem appears to be new. Its similarity to the corresponding equation for maps to the sphere is precisely what makes it so useful: it immediately opens up the possibility of applying some of the techniques developed for the case of the sphere to the present setting too.

By taking the stereographic projection from the north pole $f(r) = (u_1(r) + iu_2(r))/(1 + u_3(r))$ we determine the stereographic representation of the problem,

$$f_t = (\alpha + \beta i) \left(f_{rr} - \frac{2\overline{f} f_r^2}{1 + |f|^2} + \frac{2n-1}{r} f_r - \frac{2n-1}{r^2} f + \frac{1}{r^2}\frac{2|f|^2 f}{1 + |f|^2} \right), \tag{2.1.8}$$

where the function here is a radial map $f : \mathbb{R}^{2n} \to \mathbb{C}$. From this representation we see right away that the harmonic maps – that is, the stationary solutions – are given explicitly in this context by $f(r) = \gamma r$ for any $\gamma \in \mathbb{C}$. In the terms of the sphere coordinates, the harmonic maps are given by a type of stereographic projection

$$u(r) = \frac{1}{1 + |\gamma|^2 r^2} \left(2\mathrm{Re}\,(\gamma) r, 2\mathrm{Im}\,(\gamma) r, 1 - |\gamma|^2 r^2 \right). \tag{2.1.9}$$

Again, this is consistent with the $n = 1$ case, where the equivariant harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ are known to be stereographic projections. It is remarkable that the analytic expressions for the harmonic maps are independent of n. This seems to suggest that, from the perspective of the theory of harmonic maps, $\mathbb{CP}^n$ is the natural higher dimensional analog of $\mathbb{S}^2$.

However there is a difference for $n \geq 2$: observe that from (2.1.9) we have $\lim_{r\to\infty} u(r) = -e_3$, and so we see, by previous remarks on the energy, that in this equivariant context all of the non-trivial harmonic maps have infinite energy.

2.1.1.2 · Self-similar solutions

We first study the self similar solutions of the problem, which are given by $u(r,t) = \psi(r/\sqrt{t})$ for a profile $\psi(r) = u(r,1)$. By substituting this ansatz into (2.1.7) we determine the following ODE system on ψ:

$$\begin{aligned} 0 &= (\alpha P + \beta u\times)\left(\frac{\partial^2\psi}{\partial r} + \left(\frac{2n-1}{r} + \frac{r}{2}\right)\frac{\partial\psi}{\partial r} + \frac{2n-2+\psi_3}{r^2}e_3\right), \qquad (2.1.10) \\ \psi(0) &= e_3, \\ \psi'(0) &= v = (v_1, v_2, 0) \in T_{e_3}\mathbb{S}^2. \end{aligned}$$

As mentioned previously, the self similar solutions for the Schrödinger maps equation in this equivariant setting have already been studied in [16]. However by using the sphere representation we are able to simplify and extend the analysis, and we also treat the GLL equation as well as the Schrödinger maps equation.

Theorem 2.1.1. *Fix $\alpha \geq 0$ and $\beta \in \mathbb{R}$. For every $v \in T_{e_3}\mathbb{S}^2$ there is a unique global solution to (2.1.10). The solution is smooth for $r > 0$. In the non-trivial case, when $v \neq 0$, the solution has the following properties:*

1. *For all $r > 0$, $\psi(r) \neq e_3$.*
2. *If $\alpha > 0$ then $|\psi_r| \lesssim 1/r^3$. If $\alpha = 0$ then $r|\psi_r| \to 0$ as $r \to \infty$.*
3. *If $v \neq 0$, there exists a point $\psi_\infty \in \mathbb{S}^2$, $\psi_\infty \neq e_3$, such that $\lim_{r\to\infty}\psi(r) = \psi_\infty$. Consequently, $\mathcal{E}(\psi) = \infty$.*
4. *The limit ψ_∞ depends continuously on v; in particular, $\lim_{v\to 0}\psi_\infty = e_3$.*

Because of the convergence, we see that $u(r,t) = \psi(r/\sqrt{t})$ is a solution of the GLL flow corresponding to the initial data $u(r,0) \equiv \psi_\infty$.

Notice that in the case $\alpha > 0$ – that is, when there is some heat flow – we are able to prove faster convergence to 0 of ψ_r. In the case of the Schrödinger maps equation the rate of convergence of ψ_r is insufficient to guarantee the convergence of ψ, so an additional argument is needed.

2.1.1.3 · Global critical wellposedness

In this section we illustrate how methods for proving wellposedness of the Schrödinger maps equation for the sphere may be adapted to prove wellposedness of the equivariant Schrödinger maps equation for complex projective space. We specifically adapt the Hasimoto transformation method from [14]. For a solution $u(r,t)$ of (2.1.7) and a fixed time t, the map $r \mapsto u(r,t)$ defines a curve on $\mathbb{S}^2$ starting at e_3. Choose any element $e \in T_{e_3}\mathbb{S}^2$ and consider the parallel transport $e(r)$ of this curve along $r \mapsto u(r,t)$. Because the tangent space at the point $u(r,t)$ of the sphere is two-dimensional, it is spanned by $e(r)$ and $Je(r) = u \times e(r)$. We may therefore define a complex valued function q by the formula,

$$\mathrm{Re}\,(q)e(r) + \mathrm{Im}\,(q)Je(r) = qe(r) = u_r, \qquad (2.1.11)$$

precisely as in [14]. This equation is known as the *Hasimoto transformation*. It is chosen so that the function q will satisfy a 'nice' nonlinear Schrödinger equation; namely, an equation where the non-linearity does not contain derivatives. We derive the equation on q for all n, and in the case $n = 2$ – that is, for the equivariant Schrödinger maps equation from $\mathbb{C}^2$ to $\mathbb{CP}^2$ – we provide the necessary estimates to prove the following small-data critical global wellposedness result.

Theorem 2.1.2. *Fix $p \in [1,2]$. Define r by $1/r = 1/2 - 1/6p$ and the spaces X and X_0 by the norms,*

$$\|q\|_X = \|\nabla q\|_{L_t^{3p} L_x^r} \quad \text{and} \quad \|q\|_{X_0} = \|e^{it\Delta}(aq)\|_X,$$

where $a(x) = x_1/r$. There exists $\epsilon > 0$ such that if $u_0 : \mathbb{R}^{2n} \to \mathbb{S}^2$ is radial, q_0 is defined by (2.1.11)*, and $\|q_0\|_{X_0} \leq \epsilon$, there is unique global solution of the Schrödinger maps equation* (2.1.7) *for $n = 2$ with the derivate term q in the space X.*

Some remarks.

- This is, to the best of our knowledge, the first global wellposedness result for the Schrödinger maps equation where the target manifold has complex dimension greater than one.
- The space X is at the scaling level of the equation.
- Because $(3p, r)$ is an admissible exponent pair for the Strichartz estimates for the Schrödinger equation, we have $\|q\|_{X_0} \lesssim \|\nabla(aq)\|_{L^2} \lesssim \|\nabla q\|_{L^2}$ and hence data q_0 whose derivative is small in L^2 is included in the wellposedness result.
- For $n > 2$ we are unable to provide the estimates to close the argument in an elementary way. It would be very satisfactory to adapt this method, or another method used to prove wellposedness of Schrödinger maps from the sphere, to the present context for all n.

2.1.1.4 · The 'real' heat flow case

In the final part we discuss an interesting sub-problem of the general dynamics (2.1.7) in the special case of the harmonic map heat flow with a specific class of initial data. Recall that for the linear heat equation, if one starts with real valued data then the solution will be real valued for all time. On the other hand, if one starts the linear Schrödinger equation with real valued data then the solution will, in general, be complex valued for future times. This shows that in the heat flow case there is a lower dimensional sub-problem when one restricts to real valued data.

In our context, the analogous result is that if one starts the harmonic map heat flow with initial data valued in a great circle passing through the north pole, the solution will continue to be valued on the same great circle for future times. For the GLL flow this is not true: the solution will spread out to the whole sphere. For the harmonic map heat flow one can thus fix a great circle and consider the problem for initial data valued on that circle. One expects the analysis of this sub-problem to be easier as the dimension of the problem is reduced. However, because both the harmonic maps and the self similar solutions are solutions of this type, it is still an interesting case to consider.

By paramaterizing the great circle by its spherical distance from the north pole, one finds that the 'real' heat flow is given by the PDE,

$$g_t = \frac{\partial^2 g}{\partial r^2} + \frac{2n-1}{r}\frac{\partial g}{\partial r} + \frac{\eta(g)}{r^2}, \tag{2.1.12}$$

where $\eta(g) = \sin(2g) + (2n-2)\sin(g)$. Equations of this type, which arise in the study of the equivariant harmonic map heat flow on spherically symmetric manifolds, have been extensively studied [26, 23]. There is a general theorem which, based on the structure of η, classifies the PDE into a uniqueness regime or a non-uniqueness regime. Our primary purpose here is to show that for $n = 2$ – that is, the problem of maps from $\mathbb{C}^2$ to $\mathbb{CP}^2$ – the PDE (2.1.12) is a borderline case for this classification theorem. We find that the dynamics of the PDE share some of features of the uniqueness regime, and some of the features of the non-uniqueness regime, but ultimately that non-uniqueness holds.

Theorem 2.1.3. *(i) For $n = 2$ there is a weak non-constant solution of* (2.1.12) *corresponding to the initial data $g_0(r) \equiv \pi$. This solution is distinct from the constant solution $g(r,t) \equiv \pi$.*

(ii) In the case $n \geq 3$, for each initial data in L^∞ and each $T > 0$, there is at most one solution of (2.1.12) *in $L^\infty([0,T], L^\infty)$.*

§2.2 · THE EQUIVARIANT ANSATZ AND DERIVATION OF THE EQUATION

2.2.1 · THE EQUIVARIANT ANSATZ

We consider maps $v : \mathbb{C}^n \to \mathbb{CP}^n$. In order to rigorously describe the equivariant ansatz, we recall more carefully the construction of $\mathbb{CP}^n$. One begins with vectors $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \backslash \{0\}$ and first identifies points $z \sim \lambda z$ where $\lambda \in \mathbb{R} \backslash \{0\}$. The resulting equivalence classes can be identified with points on the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. This sphere has the usual metric induced from $\mathbb{C}^{n+1}$. Now one defines the equivalence relation $z \sim e^{i\theta} z$ for $\theta \in \mathbb{R}$, and defines $\mathbb{CP}^n = \mathbb{S}^{2n+1}/\sim$. The Fubini-Study metric is the metric induced from $\mathbb{S}^{2n+1}$.

To make the equivariant ansatz, we first construct a special class of isometries on $\mathbb{CP}^n$ in the following way. Take any complex isometry A of $\mathbb{C}^n$, and define $\hat{A} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ by,

$$\hat{A}(z_0, z_1, \ldots, z_n) = (z_0, A(z_1, \ldots, z_n));$$

that is, A acts on the last n coordinates of a point in $\mathbb{C}^{n+1}$. If A is a complex isometry of $\mathbb{C}^n$, then $\hat{A}$ is clearly a complex isometry of $\mathbb{C}^{n+1}$. Now define a map $\tilde{A}$ on $\mathbb{CP}^n$ through the homogeneous coordinates by,

$$\tilde{A}[z_0, z_1, \ldots, z_n] = [\hat{A}(z_0, z_1, \ldots, z_n)] = [z_0, A(z_1, \ldots, z_n)]. \quad (2.2.1)$$

The map $\tilde{A}$ is well defined because A commutes with complex scalar multiplication. It is an isometry of $\mathbb{CP}^n$ by the following Lemma.

Lemma 2.2.1. *Any complex isometry $\hat{B}$ of $\mathbb{C}^{n+1}$ induces an isometry $\tilde{B}$ of $\mathbb{CP}^n$ given in the homogeneous coordinates by $\tilde{B}[z] = [\hat{B}z]$ for $z \in \mathbb{C}^{n+1}$. In particular, if A is a complex isometry of $\mathbb{C}^n$ then $\tilde{A}$ defined by* (2.2.1) *is an isometry of $\mathbb{CP}^n$.*

Proof. We have,

$$\begin{aligned} d_{\mathbb{CP}^n}(\tilde{B}[v], \tilde{B}[w]) &= d_{\mathbb{CP}^n}([\hat{B}v], [\hat{B}w]) \\ &= \min_{\alpha,\beta\in[0,2\pi]} d_{\mathbb{S}^{2n+1}}(e^{i\alpha}\hat{B}v, e^{i\beta}\hat{B}w) \\ &= \min_{\alpha,\beta\in[0,2\pi]} 2\arcsin\left(\frac{1}{2}d_{\mathbb{C}^{n+1}}(e^{i\alpha}\hat{B}v, e^{i\beta}\hat{B}w)\right) \\ &= \min_{\alpha,\beta\in[0,2\pi]} 2\arcsin\left(\frac{1}{2}d_{\mathbb{C}^{n+1}}(e^{i\alpha}v, e^{i\beta}w)\right) = d_{\mathbb{CP}^n}([v],[w]), \end{aligned}$$

where in the second to last equality we used that $\hat{B}$ commutes with $e^{i\theta}$ and that $\hat{B}$ is an isometry of $\mathbb{C}^{n+1}$. □

We say a map $v : \mathbb{C}^n \to \mathbb{CP}^n$ is *equivariant* if $v(Az) = \tilde{A}v(z)$ for all complex isometries A of $\mathbb{C}^n$. We now show that this assumption implies a strong rigidity on v. Take any $z \in \mathbb{C}^n$ and write $v(z) = [w_0, w]$ for some $w_0 \in \mathbb{C}$ and $w \in \mathbb{C}^n$. Now consider any isometry A that fixes z. By the equivariant ansatz and $Az = z$ we have,

$$[w_0, Aw] = \tilde{A}u(z) = u(Az) = u(z) = [w_0, w],$$

which implies that $Aw = w$, so A also fixes w. Because A is an arbitrary isometry that fixes z, we must in fact have $w = f(z)z$ for some $f(z) \in \mathbb{C}$, and hence $v(z) = [w_0, f(z)z]$ for all z. Moreover, we have,

$$[w_0, f(Az)Az] = v(Az) = \tilde{A}v(z) = [w_0, A(f(z)z)] = [w_0, f(z)Az],$$

so $f(Az) = f(z)$. Because this holds for all isometries A, $f(z)$ is in fact a radial function and hence,

$$v(z) = [w_0, f(|z|)z], \tag{2.2.2}$$

for some function $f : \mathbb{R}^+ \to \mathbb{C}$.

We now observe that if $r \in \mathbb{R}^+$ then $v(re_1) = [w_0, f(r)r, 0, \ldots, 0]$. In other words,

$$v(\mathbb{R}^+e_1) \subset \{[w_0, w_1, 0, \ldots, 0] \ : \ w_0, w_1 \in \mathbb{C}\} \simeq \mathbb{CP}^1.$$

The Fubini-Study metric on $\mathbb{CP}^n$ restricts to the Fubini-Study metric on $\mathbb{CP}^1$, and so this $\mathbb{CP}^1$ is isometric to the sphere $\mathbb{S}^2$. Moreover, the complex structure of $\mathbb{CP}^n$ restricts to the standard complex structure of $\mathbb{CP}^1$. In the usual embedding $\mathbb{S}^2 \subset \mathbb{R}^3$ this is given, as is well known, by $Jw = u \times w$ at the point $u \in \mathbb{S}^2$ and for all $w \in T_u\mathbb{S}^2$. We next parameterize this sphere and determine an equation for the function $r \mapsto v(re_1) \in \mathbb{S}^2$.

2.2.2 · Derivation of the energy

The isometric identification between $\mathbb{CP}^1$ (with the Fubini-Study metric) and $\mathbb{S}^2 \subset \mathbb{R}^3$ (with the metric from the standard embedding) can be made through the isometric invertible map,

$$\mathbb{S}^2 \ni (a_1, a_2, a_3) \mapsto \frac{1}{\sqrt{2}(1+a_3)^{1/2}}[1 + a_3, a_1 + ia_2] \in \mathbb{CP}^1, \tag{2.2.3}$$

where in this case the north pole $e_3 = (0, 0, 1)$ is mapped to the point $[1, 0] \in \mathbb{CP}^1$. In this identification the complex structure on $\mathbb{CP}^1$ is mapped to the standard complex structure on the sphere. Given an equivariant map v :

$\mathbb{C}^n \to \mathbb{CP}^n$, we wish to write it in a form so that $v(re_1) \in \mathbb{CP}^1$ has the representation $[1+a_3, a_1+ia_2, 0, \dots, 0]$. In fact, we can write v in the form,

$$v(z) = \frac{1}{\sqrt{2}(1+u_3)^{1/2}} \left[1+u_3, (u_1+iu_2)\frac{z}{r}\right], \tag{2.2.4}$$

for $u(r) = (u_1(r), u_2(r), u_3(r))$ satisfying $|u|_{\mathbb{R}^3} = 1$. When we substitute $z = re_1$ we recover essentially the representation in (2.2.3), and hence u parameterises the sphere in the correct, isometric, way.

(To see that $v(z) = [w_0, f(r)z]$ in (2.2.2) can be written as in (2.2.4), observe that by scaling we can assume that $(w_0, g(r)z) \in \mathbb{S}^{2n+1}$, which means $|w_0|^2 + |g(r)|^2 r^2 = 1$. We can also assume by scaling that $w_0 > 0$. This means, in fact, that $w_0 \in [0,1]$, and hence there is a unique $u_3(r) \in [-1,1]$ such that $\sqrt{2}(1+u_3(r))^{1/2} = w_0$. We then define $u_1 + iu_2 = rg(r)\sqrt{2}(1-u_3)^{1/2}$, and substuting this in gives the representation above. The condition $|w_0|^2 + |g(r)r|^2 = 1$ translates into $|u|_{\mathbb{R}^3} = 1$.)

Proposition 2.2.2. *The energy is given in the u coordinates by,*

$$\mathcal{E}(v) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |dv|^2 dx = \frac{1}{2}\int_{\mathbb{R}^{2n}} \left[|u_r|^2 + \frac{1}{r^2}\left[1-u_3^2 + 2(2n-2)(1-u_3)\right]\right] dx. \tag{2.2.5}$$

Proof. In order to calculate the energy density $|dv|^2$ of $v(z)$ we have to fix a basis for $T_z\mathbb{C}^n$, which will be $2n$ dimensional, and calculate first derivatives of v with respect to this basis. For concreteness we view v as being valued in the sphere $\mathbb{S}^{2n+1}$,

$$v(z) = \frac{1}{\sqrt{2}(1+u_3)^{1/2}} \left(1+u_3, (u_1+iu_2)\frac{z}{r}\right) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}, \tag{2.2.6}$$

and perform the computation there. The only adjustment needed to be made is as follows. Given a point $p \in \mathbb{S}^{2n+1}$, all points $e^{i\theta}p$ are mapped to the same point $[p] \in \mathbb{CP}^n$. By differentiating with respect to θ, it is apparent that in $T_p\mathbb{S}^{2n+1}$ the tangent direction $ip \in T_p\mathbb{S}^{2n+1}$ is contracted under the identification $p \sim e^{i\theta}p$. Hence when calculating derivatives at the level of $\mathbb{S}^{2n+1}$ we take usual Euclidean derivatives in $\mathbb{C}^{n+1}$, project onto $T_p\mathbb{S}^{2n+1}$, and then factor out the real subspace spanned by ip. In fact, that the last two parts of this process amount to taking the complex projection,

$$Pw = w - \langle w, p\rangle_{\mathbb{C}^{n+1}} p, \tag{2.2.7}$$

of derivative terms w. We have, of course, $|Pv|^2 = |v|^2 - |\langle v, p\rangle|^2$.

Let $\partial/\partial z_k$ and $\partial/\partial \overline{z_k}$ be the usual basis for $T_z\mathbb{C}^n$. For any vector $w_0 \in \mathbb{C}^n$ define $\partial/\partial w_0 = \sum_{m=1}^n w_0^m \partial/\partial z_m$ and $\partial/\partial \overline{w_0} = \sum_{m=1}^n \overline{w_0^m}\partial/\partial\overline{z_m}$. If $\{w_k\}_{k=1}^n$ is an orthonormal basis of $\mathbb{C}^n$ then the derivatives $\{\partial/\partial w_k, \partial/\partial\overline{w_k}\}$ are an orthogonal basis for the tangent space and so, by the expression for $|dv|^2$ local in coordinates,

$$|dv|^2 = 4\sum_{k=1}^n \left|P\frac{\partial v}{\partial w_k}\right|^2 + \left|P\frac{\partial v}{\partial \overline{w_k}}\right|^2. \tag{2.2.8}$$

One verifies the formulas at the point $z \in \mathbb{C}^n$,

$$\frac{\partial r}{\partial w_0} = \frac{\langle w_0, z\rangle}{2r}; \qquad \frac{\partial r}{\partial \overline{w}} = \frac{\langle z, w_0\rangle}{2r}; \qquad \frac{\partial z}{\partial w_0} = \frac{w_0}{|w_0|}; \qquad \frac{\partial z}{\partial \overline{w_0}} = 0. \tag{2.2.9}$$

We then set $w_1 = z/|z|$ and define $w_2(z), \dots, w_n(z)$ locally so that that $\{w_k(z)\}_{k=1}^n$ is an orthonormal basis of $\mathbb{C}^n$ for each z. In this setup, w_1 is the radial direction and w_k derivatives for $k \geq 2$ will be independent of radial terms.

Hence for $k \geq 2$ we compute and find,

$$\frac{\partial v}{\partial w_k} = \frac{1}{\sqrt{2}(1+u_3)^{1/2}} \left(0, (u_1 + i u_2)\frac{w_k}{r}\right) \quad \text{and} \quad \frac{\partial v}{\partial \overline{w_k}} = 0.$$

We see from (2.2.6) that $\partial v / \partial w_k$ is complex orthogonal to v and so,

$$\left| P \frac{\partial v}{\partial w_k} \right|^2 = \left| \frac{\partial v}{\partial w_k} \right|^2 = \frac{1}{2(1+u_3)} \frac{u_1^2 + u_2^2}{r^2} = \frac{1-u_3}{2r^2},$$

where in the step we used $u_1^2 + u_2^2 + u_3^2 = 1$.

We now differentiate with respect to w_1 and $\overline{w_1}$. In this case the radial terms will also be differentiated. We note, however, that when differentiating that we can ignore the scaling term $1/(\sqrt{2}(1+u_3)^{1/2})$: when this is differentiated we simply get a scalar multiple of $v(z)$, which disappears under the projection (2.2.7). Hence,

$$\begin{aligned} P \frac{\partial v}{\partial w^1} &= \frac{1}{\sqrt{2}(1+u_3)^{1/2}} P \frac{\partial}{\partial w^1} \left[1 + u_3, (u_1 + i u_2)\frac{z}{r} \right] \\ &= \frac{1}{\sqrt{2}(1+u_3)^{1/2}} P \left(u_3' \frac{1}{2}, (u_1' + i u_2')\frac{z}{2r} - (u_1 + i u_2)\frac{z}{2r^2} \right), \end{aligned}$$

and similarly,

$$P \frac{\partial v}{\partial \overline{w^1}} = \frac{1}{\sqrt{2}(1+u_3)^{1/2}} P \left(u_3' \frac{1}{2}, (u_1' + i u_2')\frac{z}{2r} + (u_1 + i u_2)\frac{z}{2r^2} \right).$$

The difference in sign gives rise to the simplification,

$$\left| P \frac{\partial v}{\partial w^1} \right|^2 + \left| P \frac{\partial v}{\partial \overline{w^1}} \right|^2 = \frac{1}{4(1+u_3)} \left[\left| P \left(u_3', (u_1' + i u_2')\frac{z}{r} \right) \right|^2 + \left| P \left(0, (u_1 + i u_2)\frac{z}{r^2} \right) \right|^2 \right].$$

Finally, a computation using the relations $u_1^2 + u_2^2 + u_3^2 = 1$ and $u_1 u_1' + u_2 u_2' + u_3 u_3' = 0$ reveals that,

$$\begin{aligned} \left| P \left(u_3', (u_1' + i u_2')\frac{z}{r} \right) \right|^2 &= \left| \left(u_3', (u_1' + i u_2')\frac{z}{r} \right) \right|^2 \\ &\quad - \left| \left\langle \left(u_3', (u_1' + i u_2')\frac{z}{r} \right), \frac{1}{\sqrt{2}(1+u_3)^{1/2}} \left(1 + u_3, (u_1 + i u_2)\frac{z}{r} \right) \right\rangle \right|^2 \\ &= |u_r|^2 - \frac{1}{2(1+u_3)} \left| (1+u_3) u_3' + (u_1 - i u_2)(u_1' + i u_2') \right|^2 \\ &= |u_r|^2 - \frac{1}{2(1+u_3)} \left| u_3' + i(u_1 u_2' - u_1' u_2) \right|^2 \\ &= |u_r|^2 - \frac{1}{2(1+u_3)} \left[(u_3')^2 + u_1^2 (u_2')^2 + (u_1')^2 u_2^2 - 2 u_1 u_1' u_2 u_2' \right] \\ &= |u_r|^2 - \frac{1}{2(1+u_3)} \left[(1 - u_3^2) |u_r|^2 \right] = \frac{1+u_3}{2} |u_r|^2, \end{aligned}$$

and,

$$\left|P\left(0,(u_1+iu_2)\frac{z}{r^2}\right)\right|^2 = \left|P\left(\frac{-1-u_3}{r},0\right)\right|^2$$
$$= \frac{1}{r^2}\left[(1+u_3)^2 - \frac{1}{2(1+u_3)}(1+u_3)^4\right] = \frac{(1+u_3)(1-u_3^2)}{2r^2}.$$

We have, then, by substituting these expressions into (2.2.8),

$$|du|^2 = \frac{|u_r|^2}{2} + \frac{1}{r^2}\left[\frac{1-u_3^2}{2} + 2(n-1)(1-u_3)\right],$$

and then,

$$\mathcal{E}(v) = \int_{\mathbb{R}^{2n}} |dv|^2 dx = \frac{1}{2}\int_{\mathbb{R}^{2n}}\left[|u_r|^2 + \frac{1}{r^2}\left[1-u_3^2+2(2n-2)(1-u_3)\right]\right]dx,$$

which completes the computation. □

By the relations $1-u_3^2 = u_1^2+u_2^2$ and,

$$|u-e_3|^2 = u_1^2+u_2^2+(u_3-1)^2 = 2(1-u_3),$$

we can equivalently write the energy as in an L^2 form as,

$$\mathcal{E}(v) = \frac{1}{2}\int_{\mathbb{R}^{2n}}\left[|u_r|^2 + \frac{1}{r^2}\left[u_1^2+u_2^2+(2n-2)|u-e_3|^2\right]\right]dx. \tag{2.2.10}$$

With this representation we determine the following result.

Proposition 2.2.3. *There holds* $\|u_r\|_{L^2}^2 \lesssim \mathcal{E}(u) \lesssim \|u_r\|_{L^2}^2$.

Proof. The lower bound is obvious. For the upper bound, we observe that $u_1^2+u_2^2 \le |u-e_3|^2$ and hence that,

$$\mathcal{E}(u) \le \frac{1}{2}\left(\|u_r\|_{L^2}^2 + (2n-1)\left\|\frac{u-e_3}{r}\right\|_{L^2}^2\right),$$

and the result follows from the Hardy inequality $\|\phi/r\|_{L^2} \lesssim \|\phi_r\|_{L^2}$ for functions $\phi : \mathbb{R}^d \to \mathbb{R}^3$ (see Theorem 2.5.6 in the Appendix). □

2.2.3 · Variation of the energy, and the flow PDEs

In order to find the PDEs corresponding to the harmonic map heat flow, the Schrödinger maps equation, and the GLL equation, we need to calculate the variation of the energy, given by the formula,

$$\int_{\mathbb{R}^{2n}} \langle \tau(u), w\rangle_{T_u\mathbb{S}^2} dx = -\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{E}(u+\epsilon w),$$

for all radial maps $w : \mathbb{R}^{2n} \to\in T\mathbb{S}^2$ such that $w(r) \in T_u\mathbb{S}^2$.

Proposition 2.2.4. *We have*

$$\tau(u) = P_{T_u\mathbb{S}^2}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2}e_3\right).$$

Proof. Using the representation (2.2.5) we find for $w \in T_u\mathbb{S}^2$,

$$\begin{aligned}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{E}(u+\epsilon w) &= \frac{1}{2}\int_{\mathbb{R}^{2n}} 2\langle u_r, w_r\rangle_{\mathbb{R}^3} + \frac{1}{r^2}\left[-2u_3 w_3 + 2(2n-2)(-w_3)\right)] \\ &= -\int_{\mathbb{R}^{2n}} \left\langle u_{rr} + \frac{2n-1}{r}u_r, w\right\rangle_{\mathbb{R}^3} + \frac{1}{r^2}\left\langle (2n-2+u_3)e_3, w\right\rangle_{\mathbb{R}^3} dx \\ &= -\int_{\mathbb{R}^{2n}} \left\langle P_{T_u\mathbb{S}^2}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2}e_3\right), w\right\rangle_{T_u\mathbb{S}^2} dx,\end{aligned}$$

and the formula follows. □

In general the harmonic map heat flow is given by $u_t = \tau(u)$, the Schrödinger maps equation is given by $u_t = J\tau(u)$, where J is the complex structure on the target, and the GLL equation is given by $u_t = (\alpha + \beta J)\tau(u)$ for $\alpha \geq 0$ and $\beta \in \mathbb{R}$. By the previous proposition, $\tau(u)$ is determined, while as discussed above, the complex structure in the u coordinates is precisely the usual complex structure on the sphere. We are therefore ready to write down the flow PDEs.

Definition 2.2.1. The equivariant generalized Landau-Lifshitz (GLL) equation from $\mathbb{C}^n$ to $\mathbb{CP}^n$ is the Cauchy problem for $u : \mathbb{R}^{2n} \to \mathbb{S}^2$ given by,

$$\begin{aligned} u_t(r,t) &= (\alpha P + \beta u\times)\left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2}e_3\right), \\ u(r,0) &= u_0(r), \text{ with } u_0(0) = 0, \end{aligned} \tag{2.2.11}$$

for $\alpha \geq 0$ and $\beta \in \mathbb{R}$. The case $\alpha = 1$ and $\beta = 0$ is the harmonic map heat flow. The case $\alpha = 0$ and $\beta = 1$ is the Schrödinger maps equation.

Note by re-scaling time we can always assume that $\alpha^2 + \beta^2 = 1$, which we do from now on.

By taking the stereographic projection $f(r) = (u_1 + iu_2)/(1+u_3)$, with inverse given by,

$$(u_1, u_2, u_3) = \frac{1}{1+|f|^2}(2\operatorname{Re} f, 2\operatorname{Im} f, 1-|f|^2), \tag{2.2.12}$$

we can determine the *stereographic representation* of the problem. With this stereographic projection, the north pole is mapped to the origin.

Proposition 2.2.5. *The GLL equation is given in the stereographic coordinates by,*

$$f_t = (\alpha + i\beta)\left[f_{rr} - \frac{2\overline{f}f_r^2}{1+|f|^2} + \frac{2n-1}{r}f_r - \frac{2n-1}{r^2}f + \frac{1}{r^2}\frac{2|f|^2 f}{1+|f|^2}\right]. \tag{2.2.13}$$

The proof involves substituting the expression for the stereographic projection (2.2.12) into the PDE (2.2.11) and computing; we omit the computation.

The equivariant harmonic maps from $\mathbb{C}^n$ to $\mathbb{CP}^n$ are the time independent solutions of (2.2.11). Because the PDE has one space dimension, the time independent problem is an ODE. In all, ϕ is harmonic if and only if,

$$0 = \phi \times \left(\frac{d^2\phi}{dr^2} + \frac{2n-1}{r}\frac{d\phi}{dr} + \frac{(2n-2)+\phi_3}{r^2} e_3 \right), \tag{2.2.14}$$

with the boundary conditions given by $\phi(0) = e_3$ and $\phi'(0) = v = (v_1, v_2, 0) \in T_{e_3}\mathbb{S}^2$. Writing the harmonic function ϕ in the stereographic coordinates as g, the ODE is,

$$0 = g_{rr} - \frac{2\overline{g}g_r^2}{1+|g|^2} + \frac{2n-1}{r^2} g_r - \frac{2n-1}{r^2} g + \frac{1}{r^2}\frac{2|g|^2 g}{1+|g|^2},$$

and the boundary conditions are $g(0) = 0$ and $g_r(0) = v_1 + iv_2$. Remarkably, we can solve this ODE explicitly with the linear function $g(r) = (v_1 + iv_2)r$. Moreover, because it is an ODE for which we have a uniqueness theory, $g(r) = (v_1 + iv_2)r$ is the unique solution. (See the Theorem 2.5.8 in the Appendix for a local well-posedness theory for ODE of this type.) Using the stereographic projection we can write the harmonic map in the sphere coordinates as,

$$\phi(r) = \frac{1}{1+|v|^2r^2}\left(2rv_1, 2rv_2, 1-|v|^2r^2\right) = \frac{1}{1+|v|^2r^2}\left(2rv + (1-|v|^2r^2)e_3\right); \tag{2.2.15}$$

in fact, ϕ is just a version of the stereographic projection itself. This is consistent with the well-known fact that the harmonic maps in the sphere ($n = 1$) case are stereographic projections; what is interesting is that when n is incremented in the ODE (2.2.14), the new terms still cancel under this expression.

Qualitatively speaking, the harmonic maps in our context are quite simple: they start, when $r = 0$, at the north pole and, as r increases, move monotonically away from the north pole, converging to the south pole in the limit $r \to \infty$. By way of comparison, in the case of equivariant harmonic maps from the d-dimensional ball B^d to $\mathbb{S}^d$ the situation is different [35]. For $3 \le d \le 6$ the harmonic maps oscillate about the south pole, while for $d \ge 7$ the harmonic maps approach the south pole monotonically, as here. In general one finds that the equivariant harmonic maps usually fall into either an oscillatory regime or a monotonic regime [27].

Finally, we note that while the expressions above for the harmonic maps are independent of n, there is a difference when $n \ge 2$. In the case of the sphere, $n = 1$, the energy of the stereographic projection is 4π. (This may be verified by substituting (2.2.15) into the energy (2.2.5) with $n = 1$, or by consulting [3].) However, for $n \ge 2$ the energy is infinite. To see this it is sufficient to observe that,

$$\lim_{r\to\infty} \phi(r) = -e_3$$

and to use the following Lemma.

Lemma 2.2.6. *Suppose that $\mathcal{E}(u) < \infty$ and $n \ge 2$. Then $\lim_{r\to\infty} u(r)$ exists and equals e_3.*

Proof. For any $r_2 > r_1 > 0$ we have,

$$\begin{aligned} |u(r_2) - u(r_1)| = \left| \int_{r_1}^{r_2} u_r(r)dr \right| &\le \left(\int_{r_1}^{r_2} |u_r|^2 r^{2n-1} dr \right)^{1/2} \left(\int_{r_1}^{r_2} \frac{1}{r^{2n-1}} dr \right)^{1/2} \\ &\le C\mathcal{E}(u) r_1^{-n+1}, \end{aligned}$$

which, because $n \geq 2$, shows that $\lim_{r\to\infty} u(r)$ exists. This means that in the energy (2.2.10), the right most term in the integrand,

$$\frac{1}{r^2}(2n-2)|u(r)-e_3|^2 r^{2n-1},$$

converges as $r \to \infty$. For the energy to be finite, the limit must be 0. As $n \geq 2$, this implies that $\lim_{r\to\infty} u(r) = e_3$. ☐

Corollary 2.2.7. *When $n \geq 2$, the equivariant harmonic maps from $\mathbb{C}^n$ to $\mathbb{CP}^n$ all have infinite energy.*

§2.3 · SELF-SIMILAR SOLUTIONS

In this section we study self-similar solutions, which are solutions of the form $u(r,t) = \psi(r/\sqrt{t})$ for a profile $\psi(r) = u(r, 1)$.

To determine a convenient equation for the profile, we take the GLL equation (2.2.11) and multiply both sides by $(\alpha u \times +\beta P)$. Using the relationship,

$$(\alpha u \times +\beta P)(\alpha P + \beta u\times) = (\alpha^2 + \beta^2)u\times = u\times,$$

(compare to $(\alpha i + \beta)(\alpha + \beta i) = i$) we may equivalently write the PDE as,

$$\alpha u \times u_t + \beta u_t = u \times \left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2} e_3 \right). \tag{2.3.1}$$

We now substitute in $u(r,t) = \psi(r/\sqrt{t})$ to determine the ODE for the profile.

Definition 2.3.1. The self-similar problem for the GLL equation is given by the ODE,

$$-\frac{r}{2}(\alpha\psi \times \psi_r + \beta\psi_r) = \psi \times \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{2n-1}{r}\frac{\partial \psi}{\partial r} + \frac{(2n-2+\psi_3)}{r^2} e_3 \right), \tag{2.3.2}$$

subject to the initial conditions $\psi(0) = 0$ and $\psi'(0) = v = (v_1, v_2, 0) \in T_{e_3}\mathbb{S}^2$.

In the following sequence of Lemmas we will prove Theorem 2.1.1, as stated on page 80 in the introduction.

Lemma 2.3.1. *For every $v = (v_1, v_2, 0) \in T_{e_3}\mathbb{S}^2$ there is a unique global solution to* (2.3.2). *For $r > 0$ this global solution is smooth and, if $v \neq 0$, satisfies $\psi(r) \neq e_3$.*

Proof. Local existence and uniqueness in a neighborhood of the singular point $r = 0$ follows from the Theorem 2.5.8 in the Appendix. For $r > 0$, the ODE (2.3.2) is smooth and local existence, uniqueness and smoothness comes from the standard ODE theory. In order to prove global existence we establish an *a priori* bound on the derivative of ψ.

Define the function $A(r) = r^2|\psi_r|^2$ We have,

$$A'(r) = 2r|\psi_r|^2 + 2r^2\psi_{rr} \cdot \psi_r. \tag{2.3.3}$$

In order to calculate $\psi_{rr} \cdot \psi_r$, we take the inner product of the ODE (2.3.2) with $\psi \times \psi_r$. Using the fact that if v or w is orthogonal to u, then $(u \times v) \cdot (u \times w) = v \cdot w$, and also the relation $v \cdot (u \times v) = 0$, we determine that,

$$-\frac{\alpha r}{2}|\psi_r|^2 = \psi_{rr} \cdot \psi_r + \frac{2n-1}{r}|\psi_r|^2 + \frac{2n-2+\psi_3}{r^2} e_3 \cdot \psi_r,$$

and hence by solving for $\psi_{rr} \cdot \psi_r$ and substituting this into (2.3.3) we find,

$$\begin{aligned} A'(r) &= 2r|\psi_r|^2 - \left(\frac{2n-1}{r} + \frac{\alpha r}{2}\right) 2r^2|\psi_r|^2 - (2n-2+\psi_3)(\psi_3)_r \\ &= -\left(\frac{2n-3}{r} + \frac{\alpha r}{2}\right) 2A(r) - \frac{d}{dr}\left[(2n-2)\psi_3 + \frac{\psi_3^2}{2}\right]. \end{aligned} \tag{2.3.4}$$

Integrating this equation gives,

$$A(r) + \int_0^r \left(\frac{2n-3}{s} + \frac{\alpha s}{2}\right) 2A(s) ds = \left[(2n-2)(1-\psi_3) + \frac{1-(\psi_3)^2}{2}\right]. \tag{2.3.5}$$

To bound $A(r)$, we observe that the integral on the left hand side is non-negative because $A(s) \geq 0$, and so the left hand side is bounded below by $A(r)$. On the other hand, we have $\psi_3 \in [-1, 1]$ and hence the right hand side is bounded above by $4n$. This then gives $A(r) \leq 4n$, and $|\psi_r| \leq 2n/r$. This proves global existence. (The constants $4n$ and $2n$ are, of course, not optimal; they are displayed merely to show that the constants may be chosen independently of ψ.)

To prove that $\psi(r) \neq e_3$ for $r > 0$ we observe that the integral on the left hand side in (2.3.5) is increasing in r. In the non-trivial case $v \neq 0$, it is strictly increasing a neighborhood of $r = 0$ because $A'(r) = r^2|\psi_r|^2 \geq \epsilon r^2$ in a neighborhood of $r = 0$. Hence in this case the integral is strictly positive for $r > 0$. Because $A(r) \geq 0$ we see that the left hand side of (2.3.5) is strictly positive and so,

$$\left[(2n-2)(1-\psi_3) + \frac{1-(\psi_3)^2}{2}\right] > 0,$$

for $r > 0$. This gives $\psi_3(r) \neq 1$, which means $\psi(r) \neq e_3$. □

Lemma 2.3.2. *If $\alpha > 0$ we have $|\psi_r| \lesssim 1/r^3$.*

Proof. Recall the bound $A(r) \leq 4n$. Using equation (2.3.4) we have,

$$\begin{aligned} A'(r) &\leq -\frac{\alpha r}{2} A(r) - (2n-2+\psi_3)(\psi_3)_r \\ &\leq -\frac{\alpha r}{2} A(r) + \left|\frac{2n-2+\psi_3}{r^{3/2}\sqrt{\alpha/2}}\right| \cdot \left|r^{3/2}\sqrt{\alpha/2}(\psi_3)_r\right| \\ &\leq -\frac{\alpha r}{2} A(r) + \frac{1}{2}\left(\frac{8n^3}{r^3\alpha} + \frac{r^3\alpha}{2}|\psi_r|^2\right) \\ &= -\frac{\alpha r}{4} A(r) + \frac{4n^3}{r^3\alpha}. \end{aligned}$$

Integrating this equation then gives $A(r) \lesssim A(1)e^{-\alpha r^2/8} + 1/r^4 \lesssim 1/r^4$ and $|\psi_r| \lesssim 1/r^3$. (The details of how this integration may be performed are given in Proposition 2.5.9 in the appendix.) □

Lemma 2.3.3. *There exists a point* $\psi_\infty \in \mathbb{S}^2$, $\psi_\infty \neq e_3$, *such that* $\lim_{r\to\infty}\psi(r) = \psi_\infty$. *We have the convergence rate inequality* $|\psi_\infty - \psi(r)| \leq 40n^2/r^2$. *The profile* ψ *has infinite energy.*

Proof. For $\alpha > 0$, the bound $|\psi_r| \lesssim 1/r^3$ implies convergence of ψ in the limit $r \to \infty$. In the case $\alpha = 0$, when there is no heat flow contribution, the decay on the derivative is less strong, and so a different argument is needed. However in the proof we consider the general case as it is useful to know that the constant in the rate of convergence equation may be chosen independently of ψ.

We first multiply the ODE (2.3.1) by $(-\alpha\psi\times+\beta P)$. We have the relations $(-\alpha\psi\times+\beta P)(\alpha\psi\times+\beta P) = (\alpha^2+\beta^2)P = P$ and $(\psi\times)(\psi\times) = -P$ (compare to $(-\alpha i+\beta)(\alpha i+\beta) = 1$ and $(i)(i) = -1$). We can thus write the equation as,

$$\begin{aligned}
-\frac{r}{2}\psi_r &= (\alpha P + \beta\psi\times)\left(\psi_{rr} + \frac{2n-1}{r}\psi_r + \frac{2n-2+\psi_3}{r^2}e_3\right),\\
&= (\alpha + \beta\psi\times)\left(\frac{1}{r^{2n-1}}\frac{\partial}{\partial r}(r^{2n-1}\psi_r) + |\psi_r|^2\psi + \frac{2n-2+\psi_3}{r^2}Pe_3\right),
\end{aligned}$$

where in the second equality we have moved the projection P inside and expanded $P\psi_{rr} = \psi_{rr} + |\psi_r|^2\psi$. We divide through by r and integrate over $[r_1, r_2]$ to determine that,

$$-\frac{1}{2}(\psi(r_2)-\psi(r_1)) = \int_{r_1}^{r_2}(\alpha+\beta\psi\times)\left(\frac{1}{r^{2n}}\frac{\partial}{\partial r}(r^{2n-1}\psi_r) + \frac{|\psi_r|^2\psi}{r} + \frac{2n-2+\psi_3}{r^3}Pe_3\right)dr.$$

Now integrating by parts in the first term yields,

$$\begin{aligned}
-\frac{1}{2}(\psi(r_2)-\psi(r_1)) = {} & \frac{[\alpha+\beta\psi(r_2)\times]\psi_r(r_2)}{r_2} - \frac{[\alpha+\beta\psi(r_1)\times]\psi_r(r_1)}{r_1}\\
& - \int_{r_1}^{r_2}(\alpha+\beta\psi\times)\left(\frac{-2n}{r^{2n+1}}r^{2n-1}\psi_r\right)dr\\
& + \int_{r_1}^{r_2}\left(\alpha\frac{|\psi_r|^2\psi}{r} + \frac{2n-2+u_3}{r^3}(\alpha P+\beta\psi\times)e_3\right)dr.
\end{aligned}$$

Now using the bounds $|\psi(r)| = 1$ and $|\psi_r(r)| \leq 2n/r$ yields,

$$\frac{1}{2}|\psi(r_2)-\psi(r_1)| \leq \frac{2n}{r_2^2} + \frac{2n}{r_1^2} + \int_{r_1}^{r_2}\frac{4n^2}{r^3}dr + \int_{r_1}^{r_2}\left(\alpha\frac{4n^2}{r^3} + \frac{2n}{r^3}\right)dr \leq \frac{20n^2}{r_1^2},$$

which implies the solution converges with the rate given in the statement of the Lemma.

To see that the limit ψ_∞ cannot be e_3 we consider equation (2.3.5) again. As discussed previously, the integral in (2.3.5) is strictly positive and non-increasing for $r > 0$. If δ denotes the value of the integral at $r = 1$ we then have, for all $r > 1$,

$$\delta \leq \int_0^r\left(\frac{2n-3}{s} + \frac{\alpha s}{2}\right)2A(s)ds \leq \left[(2n-2)(1-\psi_3(r)) + \frac{1-(\psi_3(r))^2}{2}\right].$$

We therefore have

$$\delta \leq \left[(2n-2)(1-\psi_3(\infty)) + \frac{1-(\psi_3(\infty))^2}{2}\right],$$

which gives $\psi_\infty \neq e_3$.

Because the limit is not e_3, the profile has infinite energy by Lemma 2.2.6. □

Lemma 2.3.4. *When $\alpha = 0$ we have $\lim_{r\to\infty} r|\psi_r| = 0$.*

Proof. It is sufficient to show that $\lim_{r\to\infty} A(r) = 0$. In the $\alpha = 0$ case equation (2.3.5) reads.

$$A(r) + \int_0^r \left(\frac{2n-3}{s}\right) 2A(s)ds = \left[(2n-2)(1-\psi_3) + \frac{1-(\psi_3)^2}{2}\right].$$

We know from the previous lemma that ψ_3 converges as $r \to \infty$. The integral also converges simply because it is non-decreasing; moreover, because it is bounded above (by $4n$) it converges to a real number. We then have that $A(r)$ converges as $r \to \infty$. By examining the integral, which is finite in the limit, we see that we must have $\lim_{r\to\infty} A(r) = 0$. □

Lemma 2.3.5. *The limit ψ_∞ is a continuous function of the initial data v. In particular, as $v \to 0$ we have $\psi_\infty \to 0$.*

Proof. For convenience we will denote the self-similar profile corresponding to initial data v by $\psi_v(r)$, and we will let $\psi_v(\infty)$ denote its limit as $r \to \infty$.

The ODE local existence results give that for any $r_0 > 0$ the map $v \mapsto \psi_v(r_0)$ is continuous.

We have previously established the bound, for $r_1 < r_2$,

$$|\psi_v(r_2) - \psi_v(r_1)| \le \frac{60n^2}{r_1^2},$$

This shows that the map $v \mapsto \psi_v(r)$ converges to the map $v \mapsto \psi_v(\infty)$ uniformly, and hence that the map $v \mapsto \psi_v(\infty)$ is continuous.

Finally, we note that $\psi_0(r) \equiv 0$, $\psi_0(\infty) = 0$, and so $\lim_{v\to 0} \psi_v(\infty) = 0$, by continuity. □

With this Lemma, the proof of Theorem 2.1.1 is complete.

§2.4 · GLOBAL CRITICAL WELLPOSEDNESS IN DIMENSION TWO

In this section we prove a global critical small data wellposedness theorem for the Schrödinger maps equation for equivariant maps from $\mathbb{C}^n$ to $\mathbb{CP}^n$ when $n = 2$. The equation may be written in the sphere coordinates as,

$$u_t(r,t) = u \times \left(\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2} e_3\right), \tag{2.4.1}$$

or equivalently as,

$$-u \times u_t(r,t) = \frac{\partial^2 u}{\partial r^2} + |u_r|^2 u + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{2n-2+u_3}{r^2} P_u e_3, \tag{2.4.2}$$

where $P_u e_3$ is the projection of the vector $e_3 = (0,0,1)$ onto the tangent space at u.

Our proof relies on techniques that have been developed for the Schrödinger maps equation for the sphere. Because of the structural similarity between that equation and (2.4.2), such techniques can be adapted here. We first use a form of the Hasimoto transform to determine an equation on a derivative term of u that has a simpler nonlinearity. We then formulate the fixed point argument, and determine necessary estimates on the nonlinearity for the fixed point argument to be carried through. We conclude by proving these estimates in the case $n = 2$, thereby establishing Theorem 2.

2.4.1 · Derivation of the PDE through the Hasimoto transform

The Hasimoto transform is an extensively used tool for proving wellposedness of Schrödinger maps equations when the target is the sphere or a general complex surface. In geometric terms, it arises as follows. For fixed t, a smooth solution of (2.4.1) will satisfy $u(0,t) = e_3$. The function $r \mapsto u(r,t)$ thus defines a curve in $\mathbb{S}^2$ starting at e_3 at $r = 0$. If one fixes a unit tangent vector $e(0) \in T_{e_3}\mathbb{S}^2$, one can consider the parallel transport $e(r)$ of this vector along the curve $r \mapsto u(r,t)$; the function $e(r)$ satisfies $D_r e(r) = \nabla_{u_r} e(r) = 0$. Now because the tangent space at any point is two dimensional, the vectors $e(r)$ and $Je(r)$ give a basis for the tangent space $T_{u(r)}\mathbb{S}^2$. Any derivative of u, or other element of the tangent space, can be expressed in terms of this basis. In our case, we define a complex valued function q by the formula,

$$qe = (\operatorname{Re} q + \operatorname{Im} q\, J)e = u_r. \tag{2.4.3}$$

We then determine an equation on q. The right hand side is chosen so that q will satisfy a Schrödinger equation with a non-linearity that is easier to handle than that of (2.4.2).

Lemma 2.4.1. *The function q satisfies the PDE,*

$$iq_t = q_{rr} + \frac{2n-1}{r}q_r - \frac{2n-1}{r^2}q + N(q), \tag{2.4.4}$$

where the nonlinear term $N(q)$ is given by,

$$N(q) = \frac{d}{dr}\left[-\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds\right] + \alpha q, \tag{2.4.5}$$

for a real-valued function α satisfying,

$$\alpha_r = Re\left(\overline{q}q_r + \frac{|q|^2}{r} - \overline{q}\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds\right). \tag{2.4.6}$$

Proof. First, we recall that in the embedding $\mathbb{S}^2 \subset \mathbb{R}^3$ the covariant derivative of a vector field $v(r) \in T_{u(r)}\mathbb{S}^2$ is given by $D_r v = v_r + \langle u_r, v\rangle u$, where the inner product here is the usual inner product on $\mathbb{R}^3$.

Now let p and q satisfy $pe = u_t$ and $qe = u_r$. We will determine three equations relating p, q and u.

1. Because e satisfies $D_r e = 0$ we have,

$$q_r e = D_r(qe) = D_r(u_r) = u_{rr} + |u_r|^2 u, \tag{2.4.7}$$

which is the first two terms two term in the right hand side of (2.4.2). The next term in (2.4.2) is $((2n-1)/r)qe$.

For the projection term we calculate, using $D_r e = 0$,

$$\begin{aligned}
\frac{d}{dr}\langle P_u e_3, e\rangle &= \frac{d}{dr}\langle e_3 - \langle u, e_3\rangle u, e\rangle = \langle D_r(e_3 - \langle u, e_3\rangle u), e\rangle \\
&= \left\langle \frac{d}{dr}(e_3 - \langle u, e_3\rangle u) + \langle u_r, e_3 - \langle u, e_3\rangle u\rangle u, e \right\rangle \\
&= \langle -\langle u_r, e_3\rangle u - \langle u, e_3\rangle u_r + \langle u_r, e_3\rangle u, e\rangle = -u_3\langle u_r, e\rangle = -u_3(r)\mathrm{Re}\, p(r).
\end{aligned}$$

Using the fact that $u(0,t) = e_3$, so that $P_u e_3 = 0$ at $r = 0$, we have,

$$\langle P_u e_3, e\rangle = -\int_0^r u_3(s)\mathrm{Re}\, q(s) ds. \tag{2.4.8}$$

An identical calculation for $\langle P_u e_3, Je\rangle$ gives, in total,

$$P_u e_3 = -\left(\int_0^r u_3(s)q(s)ds\right) e(r).$$

Plugging (2.4.7) and (2.4.8) into (2.4.2) then gives,

$$ip = q_r + \frac{2n-1}{r}q - \frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds. \tag{2.4.9}$$

2. From the identity $D_r u_t = D_t u_r$ we find,

$$p_r e = D_r(pe) = D_r u_t = D_t u_r = D_t(qe) = q_t e + qD_t e. \tag{2.4.10}$$

Because e is a parallel transport vector field, $|e|^2 = 1$ and so $0 = (d/dt)|e|^2 = \langle D_t e, e\rangle$. The vector $D_t e$ is thus orthogonal to e. Because the tangent space is spanned by e and Je, we must have $D_t e = \alpha Je$ for some real-valued function α. Substituting this into (2.4.10), we get $p_r e = q_t e + q\alpha Je$, or,

$$p_r = q_t + i\alpha q. \tag{2.4.11}$$

3. To determine an equation on α we use the curvature relation $D_t D_r e = D_r D_t e + R(u_t, u_t)e$ where R is the Riemann curvature tensor. On the sphere $R(v,w)z = \langle Jv, w\rangle Jz$. Therefore, using also $D_r e = 0$, we find,

$$0 = D_r(\alpha Je) + \langle Ju_t, u_r\rangle Je = \alpha_r Je + \langle pJe, qe\rangle Je,$$

which gives $\alpha_r = -\mathrm{Im}\,(p\overline{q})$. Substituting the formula for p in (2.4.9) gives equation (2.4.6).

To determine an equation only on q we differentiate (2.4.9) with respect to r, to find,

$$ip_r = q_{rr} + \frac{2n-1}{r}q_r - \frac{2n-1}{r^2}q + \frac{d}{dr}\left[-\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds,\right].$$

Substituting the expression for p_r in (2.4.11) gives equations (2.4.4) and (2.4.5). □

We recall the Theorem from the introduction.

Theorem. *Fix $p \in [1,2]$ and define,*

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{6p},$$

and the spaces X and X_0 given by the norms

$$\|q\|_X = \|\nabla q\|_{L_t^{3p} L_x^r} \quad \textit{and} \quad \|q\|_{X_0} = \|e^{it\Delta}(aq)\|_X,$$

where $a(x) = x_1/r$. There exists $\epsilon > 0$ such that if $\|q_0\|_{X_0} \le \epsilon$ there is unique global solution of (2.4.4) *for $n = 2$ in the space X.*

We begin by determining a convenient Duhamel representation for the problem. Our Duhamel representation will be valid for all n, though we carry out the wellposedness argument for $n = 2$ only. In the following we will rely heavily on the Hardy inequalities given in Theorems 2.5.6 and 2.5.15 in the appendix.

First, we absorb the linear term $-(2n-1)q/r^2$ into the Laplacian. To do this, we fix a function $a : \mathbb{S}^{2n-1} \to \mathbb{C}$ that satisfies $\Delta_{\mathbb{S}^{2n-1}} a = -(2n-1)a$. We may concretely choose $a(x) = x_1$. To see this, extend a to a function on all of $\mathbb{R}^{2n}$ by $a(x/|x|)$. On the one hand, we have,

$$\Delta_{\mathbb{R}^{2n}} (ra(x/|x|)) = \Delta_{\mathbb{R}^{2n}} (x_1) = 0.$$

Then, using the polar representation, $\Delta_{\mathbb{R}^{2n}} = \partial_{rr} + ((2n-1)/r)\partial_r + (1/r^2)\Delta_{\mathbb{S}^{2n-1}}$, we see that,

$$0 = \left[\partial_{rr} + \frac{2n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{2n-1}}\right](ra(x/|x|)) = \left[0 + \frac{2n-1}{r}\right]a(x/|x|) + \frac{r}{r^2}\Delta_{\mathbb{S}^{2n-1}} a(x/|x|),$$

and so,

$$\Delta_{\mathbb{S}^{2n-1}} a(x/|x|) = -(2n-1)a(x/|x|).$$

Now defining $w(x,t) = q(r,t)a(x/|x|)$, we see that,

$$\Delta_{\mathbb{R}^{2n}} w = \frac{\partial^2 q}{\partial r^2} a + \frac{2n-1}{r}\frac{\partial q}{\partial r} a - \frac{2n-1}{r^2} qa. \tag{2.4.12}$$

This is exactly the Laplacian term in the PDE (2.4.4) multiplied by a.

In terms of estimates, we have the pointwise estimate $|\nabla a| \le 1/r$, which is determined from a calculation. For Lebesgue estimates we have,

$$\begin{aligned}
\|w\|_{L^p}^p = \|qa\|_{L^p}^p &= \int_0^\infty \left(|q(r)|^p \int_{r\mathbb{S}^{2n-1}} |a(x/|x|)|^p dx\right) dr \\
&= \int_0^\infty \left(|q(r)|^p r^{2n-1} \int_{\mathbb{S}^{2n-1}} |a(x/|x|)|^p dx\right) dr \\
&= C\|q\|_{L^p}^p,
\end{aligned}$$

where $C = \|a\|_{L^p(\mathbb{S}^{2n-1})}/|\mathbb{S}^{2n-1}| < \infty$. We also have,

$$\|\nabla q\|_{L^p} \sim \|\nabla_r q\|_{L^p} \sim \|\nabla_r (aq)\|_{L^p} \lesssim \|\nabla_r (aq)\|_{L^p} + \|\nabla_\theta (aq)\|_{L^p} \sim \|\nabla w\|_{L^p},$$

while,

$$\|\nabla w\|_{L^p} \sim \|\nabla(a)q\|_{L^p} + \|a\nabla_r q\|_{L^p} \lesssim \|\frac{1}{r}q\|_{L^p} + \|\nabla_r q\|_{L^p} \lesssim \|\nabla q\|_{L^p},$$

so in conclusion $\|\nabla q\|_{L^p} \sim \|\nabla w\|_{L^p}$.

In the next Lemma we rather carefully verify that we can recover solutions to the PDE for q from solutions to the PDE for w.

Lemma 2.4.2. *The PDE on w is given by,*

$$w_t = \Delta w + N(q)a, \tag{2.4.13}$$

or in Duhamel form by,

$$w(x,t) = e^{it\Delta}w(x,0) + i\int_0^r e^{i(t-s)\Delta}N(q(r,s))a(x/|x|,s)ds. \tag{2.4.14}$$

If the solution $w(x,t)$ corresponding to initial data of the form $w(x,0) = q(r)a(x/|x|)$ is unique, then the solution is of the form $w(x,t) = q(r,t)a(x/|x|)$, where q satisfies (2.4.4).

Proof. To determine the equation (2.4.13) for w we simply multiply the PDE for q (2.4.4) by a, and use the expression (2.4.12) for Δw. The Duhamel representation is standard.

We now show how solutions of (2.4.4) may be recovered from solutions of the equation for w. Let w be a solution of (2.4.14) and define $\tilde{w} = -(1/(2n-1))\Delta_{\mathbb{S}^{2n-1}}w$. Assuming uniqueness we will show that $\tilde{w} = w$. We take the spherical Laplacian $-(1/(2n-1))\Delta_{\mathbb{S}^{2n-1}}$ of (2.4.13), noting that it commutes both with $\Delta = \Delta_{\mathbb{R}^{2n}}$ and $N(q)$, as $N(q)$ is radial. We then find that $\tilde{w}$ satisfies the same PDE (2.4.13) as w. Moreover, we have,

$$\tilde{w}(x,0) = -\frac{1}{2n-1}\Delta_{\mathbb{S}^{2n-1}}w(x,0) = -\frac{1}{2n-1}\Delta_{\mathbb{S}^{2n-1}}\left[q(r)a(x/|x|)\right] = w(x,0),$$

and so by uniqueness, $\tilde{w}(x,t) = -(1/(2n-1))\Delta_{\mathbb{S}^{2n-1}}w(x,t) = w(x,t)$. This means that w is a radial function times an eigenfunction of the Laplacian of the sphere of $\mathbb{S}^{2n-1}$ with eigenvalue $-(2n-1)$.

Let $T_k : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the linear map that multiplies the kth component of $x \in \mathbb{R}^{2n}$ by -1 and leaves the other components fixed. From the representation of a we see that for $k = 1$ we have $w_0(T_k x) = -w_0(x)$ while for $k \geq 2$ we have $w_0(T_k x) = w_0(x)$. By uniqueness, $x \mapsto w(x,t)$ inherits these properities also. But now the only eigenfunction of the Laplacian on the sphere with eigenvalue $-(2n-1)$ with these symmetries is precisely a. Therefore $w(x,t) = q(r,t)a(x/|x|)$. Substituting this expression into the PDE (2.4.13) for w yields the PDE (2.4.4) for q. □

By virtue of this Lemma, we can perform the fixed point argument on w. The next Lemma describes sufficient estimates for this fixed point argument to hold, and in the proof the fixed point argument is described.

2.4.3 · WELLPOSEDNESS WHEN $n = 2$

For the reminder of this section we fix $n = 2$.

Before stating the lemma we fix some index notation. In the course of the proof we will need to handle Lebesgue space norms of quantities like q, q_r, q^2, qq_r, etc., and other quantities which scale like these. We are led to define the index,

$$\frac{1}{s(i,j)} = \frac{i+j}{4} - \frac{i}{6p}. \tag{2.4.15}$$

We will put items that scale like the product of i copies of q with a total of j derivatives in the space $L_x^{s(i,j)}$. For example, we will put q in $L_x^{s(1,0)}$, we will put q^2 in $L_x^{s(2,0)}$ and qq_r in $L_x^{s(2,1)}$. In this way, critical scaling is maintained throughout as, for example, $\|qq_r\|_{L_x^{s(2,1)}}$ is invariant under scaling.

The Strichartz inequality we will use is,

$$\left\| \int_0^t e^{i(t-s)\Delta} G ds \right\|_{L_t^{3p} L_x^r} \lesssim \|G\|_{L_t^p L_x^{s(3,1)}} ;$$

this is classical: see, for example, [43]. The Hölder inequality is,

$$\|fg\|_{L_x^{s(i+k,j+m)}} \leq \|f\|_{L_x^{s(i,j)}} \cdot \|g\|_{L_x^{s(k,m)}};$$

and the Sobolev is, for $k < l$,

$$\|\nabla^k f\|_{L_x^{s(i,j)}} \lesssim \|\nabla^l f\|_{L_x^{s(i,j+l-k)}}.$$

One verifies that these inequalities hold by checking the relevant exponent conditions.

Finally, note that $s(1,1) = r$.

Lemma 2.4.3. *For Theorem 2 to be true, it is sufficient that the following bounds hold:*

$$\|\nabla N(q)\|_{L_x^{s(3,1)}} \lesssim \|\nabla q\|^3_{L_x^{s(1,1)}}, \tag{2.4.16}$$

$$\|\nabla(N(q_1) - N(q_2))\|_{L_x^{s(3,1)}} \lesssim \|\nabla(q_1 - q_2)\|_{L_x^{s(1,1)}} \left(\|\nabla q_1\|^2_{L_x^{s(1,1)}} + \|\nabla q_2\|^2_{L_x^{s(1,1)}} \right). \tag{2.4.17}$$

Proof. Well-posedness follows by a fixed point argument for the operator

$$Tw = e^{it\Delta} w(x,0) + i \int_0^t e^{i(t-s)\Delta} N(q(r,s)) a(x,s) ds.$$

We will show that T is a contraction mapping on a small ball around 0.

We first show that T maps a ball to itself. We have the bound,

$$\|Tw\|_X \leq \|w_0\|_{X_0} + \left\| \int_0^t e^{i(t-s)\Delta} \nabla(N(q)a) ds \right\|_{L_t^{3p} L_x^r} \lesssim \|w_0\|_{X_0} + \|\nabla(N(q)a)\|_{L_t^p L_x^{s(3,1)}} .$$

Considering the space norm of the integral, we have, by Hölder and Sobolev, and then conditions (2.4.16),

$$\begin{aligned}\|\nabla(N(q)a)\|_{L_x^{s(3,1)}} &\lesssim \|\nabla(N)a\|_{L_x^{s(3,1)}} + \|N\nabla a\|_{L_x^{s(3,1)}} \\ &\leq \|\nabla(N)\|_{L_x^{s(3,1)}} + \left\| \frac{1}{r} N \right\|_{L_x^{s(3,1)}} \leq \|\nabla(N)\|_{L_x^{s(3,1)}} \lesssim \|\nabla q\|^2_{L_x^{s(1,1)}} \lesssim \|\nabla w\|^3_{L_x^{s(1,1)}},\end{aligned}$$

and hence, as $r = s(1,1)$,

$$\|Tw\|_X \leq \|w_0\|_{X_0} + C\|w\|^3_X.$$

Now choose ϵ_0 so that $C\epsilon_0^2 \leq 1/2$, and let $\epsilon \leq \epsilon_0$. Then, if $\|w_0\|_{X_0} \leq \epsilon/2$ and $\|w\|_X \leq \epsilon$ we have,

$$\|Tw\|_X \leq \frac{\epsilon}{2} + C\epsilon^3 \leq \frac{\epsilon}{2} + (C\epsilon_0^2)\epsilon \leq \epsilon,$$

and so T maps every ϵ ball into itself, for ϵ sufficiently small, assuming the initial data satisfies the bound

$\|w_0\|_{X_0} \le \epsilon/2$.

We next show that T is a contraction in a sufficiently small ball around 0. Let w_1 and w_2 be two solutions, with radial parts q_1 and q_2 respectively. We have,

$$Tw_1 - Tw_2 = \int_0^t e^{i(t-s)\Delta}(\nabla(N(q_1)a) - \nabla(N(q_2)a))ds,$$

which gives, using (2.4.17),

$$\begin{aligned}
&\|\nabla(N(q_1)a) - \nabla(N(q_2)a)\|_{L_x^{s(3,1)}} \\
&\quad\lesssim \|\nabla(N(q_1) - N(q_2))a\|_{L_x^{s(3,1)}} + \|(N(q_1) - N(q_2))\nabla a\|_{L_x^{s(3,1)}} \\
&\quad\lesssim \|\nabla(N(q_1) - N(q_2))a\|_{L_x^{s(3,1)}} + \left\|\frac{1}{r}(N(q_1) - N(q_2))\right\|_{L_x^{s(3,1)}} \\
&\quad\lesssim (\|\nabla q_1\|^2_{L_x^{s(1,1)}} + \|\nabla q_2\|^2_{L_x^{s(1,1)}})\|\nabla(q_1 - q_2)\|_{L_x^{s(1,1)}}, \\
&\quad\lesssim (\|\nabla w_1\|^2_{L_x^{s(1,1)}} + \|\nabla w_2\|^2_{L_x^{s(1,1)}})\|\nabla(w_1 - w_2)\|_{L_x^{s(1,1)}},
\end{aligned}$$

and so

$$\|Tw_1 - Tw_2\|_X \lesssim (\|w_1\|_X^2 + \|w_2\|_X^2)\|w_1 - w_2\|_X,$$

and hence by choosing the ball small enough, T is a contraction. □

Lemma 2.4.4. *When $n = 2$ the bounds* (2.4.16) *and* (2.4.17) *hold.*

Proof. Write,

$$N = \frac{d}{dr}\left(-\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds\right) + \alpha q =: N_1 + N_2,$$

and recall,

$$\alpha_r = \mathrm{Re}\left(q_r\overline{q} + \frac{|q|^2}{r} - \overline{q}\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds\right). \tag{2.4.18}$$

We will prove the bounds for N_1 first.

We have,

$$\begin{aligned}
\|\nabla N_1\|_{s(3,1)} &\lesssim \left\|\nabla^2\left(-\frac{2n-2+u_3}{r^2}\int_0^r u_3(s)q(s)ds\right)\right\|_{L_x^{s(3,1)}} \\
&\lesssim \left\|(\nabla^2 u_3)\frac{1}{r^2}\int_0^r u_3(s)q(s)ds\right\|_{L_x^{s(3,1)}} \\
&\qquad + \left\|-\frac{2n-2+u_3}{r^2}\frac{1}{r^2}\int_0^r u_3(s)q(s)ds\right\|_{L_x^{s(3,1)}} \\
&\qquad + \left\|-\frac{2n-2+u_3}{r^2}\nabla(u_3(r)q(r))ds\right\|_{L_x^{s(3,1)}} \\
&=: A + B + C
\end{aligned}$$

From the equation $u_{rr} + |u_r|^2 u = q_r e$, we have $|u_{rr}| \leq |q|^2 + |q|$ pointwise. Therefore, for A,

$$\begin{aligned}
A &\leq \left\| (|q|^2 + |q_r|) \frac{1}{r^2} \int_0^r u_3(s) q(s) ds \right\|_{L_x^{s(3,1)}} \\
&= \|q\|_{L_x^{s(1,0)}} \left\| \frac{q}{r} \right\|_{L_x^{s(1,1)}} \left\| \frac{1}{r} \int_0^r u_3(s) q(s) ds \right\|_{L_x^{s(1,0)}} + \|q_r\|_{L_x^{s(1,1)}} \left\| \frac{1}{r^2} \int_0^r u_3(s) q(s) ds \right\|_{L_x^{s(2,0)}} \\
&= \|q\|_{L_x^{s(1,0)}} \|q/r\|_{L_x^{s(1,1)}} \|u_3(s) q(s)\|_{L_x^{s(1,0)}} + \|q_r\|_{L_x^{s(1,1)}} \left\| \frac{1}{r} u_3(s) q(s) ds \right\|_{L_x^{s(2,0)}} \\
&\lesssim \|q\|^2_{L_x^{s(1,0)}} \|q_r\|_{L_x^{s(1,1)}} + \|q_r\|_{s(1,1)} \left\| \frac{1}{r} u_3(r) q(r) \right\|_{L_x^{s(2,0)}} \\
&\lesssim \|q\|^2_{L_x^{s(1,0)}} \|q_r\|_{L_x^{s(1,1)}} + \|q_r\|_{s(1,1)} \left\| \frac{u_3}{r} \right\|_{L_x^{s(1,0)}} \|q(r)\|_{L_x^{s(1,0)}} \lesssim \|q_r\|^3_{L_x^{s(1,1)}}.
\end{aligned} \tag{2.4.19}$$

For B, we have,

$$\begin{aligned}
B &\lesssim \left\| \frac{2n - 2 + u_3}{r} \right\|_{L_x^{s(1,0)}} \left\| \frac{1}{r^3} \int_0^r u_3(s) q(s) ds \right\|_{L_x^{s(2,1)}} \\
&\lesssim \|u_r\|_{L_x^{s(1,0)}} \left\| \frac{1}{r^2} u_3(r) q(r) \right\|_{L_x^{s(2,1)}} \\
&\lesssim \|u_r\|_{L_x^{s(1,0)}} \left\| \frac{u_3(r)}{r} \right\|_{s(1,0)} \left\| \frac{q(r)}{r} \right\|_{L_x^{s(1,1)}} \\
&\lesssim \|u_r\|^2_{L_x^{s(1,0)}} \|q_r\|_{L_x^{s(1,1)}} \lesssim \|q_r\|^3_{L_x^{s(1,1)}}.
\end{aligned} \tag{2.4.20}$$

For C, we have,

$$\begin{aligned}
C &\lesssim \left\| \frac{2n - 2 + u_3}{r^2} \nabla (u_3(r) q(r)) ds \right\|_{L_x^{s(3,1)}} \\
&\lesssim \left\| \frac{2n - 2 + u_3}{r} \right\|_{L_x^{s(1,0)}} \left(\left\| \frac{1}{r} \nabla (u_3) q \right\|_{L_x^{s(2,1)}} + \left\| \frac{1}{r} (u_3) \nabla q \right\|_{L_x^{s(2,1)}} \right) \\
&\lesssim \|q\|_{L_x^{s(1,0)}} \left(\|\nabla u_3\|_{L_x^{s(1,0)}} \left\| \frac{q}{r} \right\|_{L_x^{s(1,1)}} + \left\| \frac{u_3}{r} \right\|_{L_x^{s(1,0)}} \|\nabla q\|_{L_x^{s(1,1)}} \right) \\
&\lesssim \|q_r\|^3_{L_x^{s(1,1)}}.
\end{aligned} \tag{2.4.21}$$

The three estimates (2.4.19), (2.4.20) and (2.4.21) together give the estimate $\|\nabla N_1\|_{L_x^{s(3,1)}} \lesssim \|q_r\|^3_{L_x^{(1,1)}}$.
As for N_2, we have,

$$\begin{aligned}
\|\nabla(\alpha q)\|_{L_x^{s(3,1)}} &\lesssim \|\alpha_r q\|_{L_x^{s(3,1)}} + \|\alpha q_r\|_{L_x^{s(3,1)}} \\
&\lesssim \|\alpha_r\|_{L_x^{s(2,1)}} \|q\|_{L_x^{s(1,0)}} + \|\alpha\|_{L_x^{s(2,0)}} \|q_r\|_{L_x^{s(1,1)}} \\
&\lesssim \|\alpha_r\|_{L_x^{s(2,1)}} \|q_r\|_{L_x^{s(1,1)}}.
\end{aligned} \tag{2.4.22}$$

Then, using the expression for α_r in (2.4.18) and the fact that $u_3 \in L^\infty$,

$$
\begin{aligned}
\|\alpha_r\|_{s(2,1)} &\lesssim \|qq_r\|_{L_x^{s(2,1)}} + \left\|\frac{q^2}{r}\right\|_{L_x^{s(2,1)}} + \left\|\frac{2n-2+u_3}{r^2} q \int_0^r u_3(s)q(s)ds\right\|_{s(2,1)} \\
&\lesssim \|q\|_{L_x^{s(1,0)}} \left(\|q_r\|_{L_x^{s(1,1)}} + \left\|\frac{q}{r}\right\|_{L_x^{s(1,1)}}\right) + \left\|\frac{q}{r}\right\|_{L_x^{s(1,1)}} \left\|\frac{1}{r}\int_0^r u_3(s)q(s)ds\right\|_{L_x^{s(1,0)}} \\
&\lesssim \|q_r\|^3_{L_x^{s(1,1)}} + \left\|\frac{q}{r}\right\|_{L_x^{s(1,1)}} \|q\|_{L_x^{s(1,0)}} \lesssim \|q_r\|^3_{L_x^{s(1,1)}}. \qquad (2.4.23)
\end{aligned}
$$

The estimates (2.4.22) and (2.4.23) give $\|\nabla N_2\|_{L_x^{s(3,1)}} \lesssim \|q_r\|^3_{L_x^{(1,1)}}$. and hence (2.4.16). The estimate (2.4.17) follows from an identical argument. □

Theorem 2.1.2 is thus established.

§2.5 · THE 'REAL' HEAT FLOW CASE

In this section we will discuss what might be termed the 'real' equivariant heat flow from $\mathbb{C}^n$ to $\mathbb{CP}^n$. In the case when $\alpha = 1$ and $\beta = 0$, that is, for the harmonic map heat flow, it is possible to make an ansatz which further reduces the problem. In terms of the spherical coordinates,

$$
\begin{aligned}
u_t &= u_{rr} + u|u_r|^2 + \frac{2n-1}{r}u_r + \frac{2n-2+u_3}{r^2}(e_3 - u\langle u, e_3\rangle), \qquad (2.5.1) \\
u(r,0) &= v(r) = (v_1(r), v_2(r), 0) \in T_{e_3}\mathbb{S}^2,
\end{aligned}
$$

this ansatz involves assuming that the initial data is valued in one great circle passing through the north pole; that is, the initial data is of of the form $c(r)e_3 + d(r)v_0$. (See Figure 2.1.) In this case for $t > 0$ the solution will continue to be valued in the same great circle. To see this, let $w = v \times e_3$ and let $a(r,t) = u(r,t) \cdot w$. By taking the inner product of equation (2.5.1) with aw we have,

$$
\begin{aligned}
aa_t &= aa_{rr} - a^2|u_r|^2 + \frac{2n-1}{r}aa_r + \frac{2n-2+u_3}{r^2}(-a^2u_3) \\
&\leq aa_{rr} + \frac{2n-1}{r}aa_r + \frac{2n-3}{r^2}a^2.
\end{aligned}
$$

By integrating this equation and using the Hardy inequality with best constant $4/d^2 = 4/(2n-2)^2$ we determine that,

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{C}^n}(a)^2dx &\leq \sigma_{2n-1}\int_0^\infty a\frac{\partial}{\partial r}(r^{2n-1}a_r)dr + (2n-3)\left\|\frac{a}{r}\right\|^2_{L^2} \\
&= -\sigma_{2n-1}\int_0^\infty (a_r)^2r^{2n-1}dr + (2n-3)\frac{4}{(2n-2)^2}\|a_r\|^2_{L^2} \leq 0,
\end{aligned}
$$

and hence $a(r,t) = 0$ for all time. The solution is therefore a linear combination of v_0 and e_3.

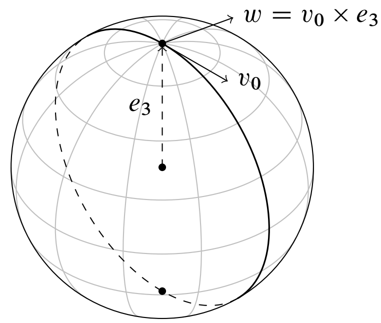

Figure 2.1: In the case of the harmonic map heat flow, if the initial data takes values in one great circle (here the great circle spanned by v_0 and e_3), then the solution will be valued in the same great circle for future times. Both the harmonic maps and the self-similar solutions constructed in section 2 are of this type.

In terms of the stereographic representation of the problem,

$$\begin{aligned} f_t &= f_{rr} - \frac{2\overline{f} f_r^2}{1+|f|^2} + \frac{2n-1}{r} f_r(r) + \frac{2n-1}{r^2} f(r) + \frac{2|f|^2 f}{1+|f(r)|^2} \\ f(r,0) &= f_0(r) \end{aligned}$$

the ansatz is that the initial data is of the form $f(r,0) = b(r)e^{i\theta}$, for some real valued function $b(r)$ and a constant θ. The solution will then be of the form $f(r,t) = b(r,t)e^{i\theta}$, for the same constant θ and for some real valued function $b(r,t)$. This motivates the terminology 'real heat flow'.

It is not surprising that this problem is simpler to analyze, and in fact with this assumption we are able to say more about the dynamics of the problem. On the other hand, this problem is still interesting because both the harmonic maps and the self-similar solutions constructed in section 2 fit into this context. In fact, the harmonic maps are given in the stereograpic coordinates by $f(r,t) = \alpha r = |\alpha| r e^{i\theta}$. The initial data for a self-similar solution is just a point, so the initial data is valued in the great circle passing through that point and the north pole.

We will now describe how, based on the ansatz just described, a simpler PDE on the solution may be determined. As the solution is valued on a great circle we can perform a change of variables,

$$u(r,t) = \cos(g)e_3 + \sin(g)v_0,$$

for an unknown real-valued g. Geometrically, g is the spherical distance between $u(r,t)$ and e_3. We calculate,

$$u_r = g_r(-\sin(g)e_3 + \cos(g)v_0),$$

and,

$$\begin{aligned} u_{rr} &= g_{rr}(-\sin(g)e_3 + \cos(g)v_0) + g_r^2(-\sin(g)e_3 - \cos(g)v_0) \\ &= g_{rr}(-\sin(g)e_3 + \cos(g)v_0) - u|u_r|^2. \end{aligned}$$

Substituting these into (2.5.1) gives,

$$g_t(-\sin(g)e_3 + \cos(g)v_0) = \left(g_{rr} + \frac{2n-1}{r}g_r\right)(-\sin(g)e_3 + \cos(g)v_0)$$
$$+ \frac{2n-2+\cos(g)}{r^2}(e_3 - \cos(g)(\cos(g)e_3 + \sin(g)v_0)).$$

Taking the inner product of this equation with $-\sin(g)e_3 + \cos(g)v_0$ then yields the equation on g.

Definition 2.5.1. The real heat flow problem is the Cauchy problem,

$$g_t = g_{rr} + \frac{2n-1}{r}g_r - \frac{1}{r^2}\left[(2n-2)\sin(g) + \frac{1}{2}\sin(2g)\right] \tag{2.5.2}$$

subject to the initial condition $g(r,0) = g_0(r)$.

For convenience we let $\eta(x) = (2n-2)\sin(x) + \sin(2x)/2$.

Definition 2.5.2. The stationary real heat flow problem is the ODE,

$$0 = \psi_\alpha''(r) + \frac{2n-1}{r}\psi_\alpha'(r) - \frac{1}{r^2}\eta(\psi_\alpha), \tag{2.5.3}$$

subject the initial conditions $\psi_\alpha(0) = 0$ and $\psi_\alpha'(0) = \alpha > 0$.

In the spherical coordinates the stationary solutions – that is, the harmonic maps – are given explicitly in (2.2.15). By transforming these solutions into the coordinates g, one finds that the unique solutions to the stationary real heat flow problem are,

$$\psi_\alpha(r) = 2\arctan(\alpha r),$$

which may be verified by substitution into (2.5.3). In light of later results, what will be most notable about the explicit solution is that it is independent of n.

2.5.1 · Uniqueness of solutions to the PDE problem in the $n \geq 3$ case

PDEs of the type,

$$u_t = u_{rr} + \frac{d-1}{r}u_r - \frac{\eta(u)}{r^2}, \tag{2.5.4}$$

with,

$$\eta(0) = \eta(\pi) = \eta(2\pi), \qquad \eta(x) > 0 \text{ for } x \in (0,\pi), \qquad \eta(x) < 0 \text{ for } x \in (\pi, 2\pi),$$

arise naturally in the study of the equivariant harmonic map heat flow from $\mathbb{R}^d$ to spherically symmetric manifolds. There is a general theorem classifying when there is uniqueness of solutions and when there is not uniqueness [23]. It states that if,

$$\eta'(\pi) < -\frac{(d-2)^2}{4}, \tag{2.5.5}$$

then there is non-uniqueness – that is, two distinct solutions with the same initial data – while if,

$$\eta'(x) \geq -\frac{(d-2)^2}{4}, \tag{2.5.6}$$

for all x then for every initial data there is at most one solution in $L_t^\infty L_x^\infty$. We offer the following new proof of the latter case.

Proposition 2.5.1. *Suppose that $\eta'(x) \geq -(d-2)^2/4$ for all x. There there is at most one solution to* (2.5.4) *in $L_t^\infty L_x^\infty$.*

Proof. First we observe that the condition (2.5.6) implies the one-sided Lipshitz inequality,

$$\frac{\eta(u)-\eta(v)}{u-v} \geq \min_{x\in[0,2\pi]} \eta'(x) \geq -\frac{(d-2)^2}{4}.$$

Now consider two solutions u and v of (2.5.4) with the same initial data u_0 and set $\phi = u - v$. We will initially assume that $u_0 \in L^2 \cap L^\infty$, and remove the L^2 condition later. Under this assumption we calculate,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 &= \frac{1}{2}\frac{d}{dt}\sigma_{d-1}\int_0^\infty |\phi(r,t)|^2 r^{d-1}dr \\
&= \sigma_{d-1}\int_0^\infty \phi\left[\phi_{rr} + \frac{2n-1}{r}\phi_r - \frac{\eta(u)-\eta(v)}{r^2}\right] r^{2n-1}dr \\
&= -\|\phi_r\|_{L^2}^2 - \sigma_{d-1}\int_0^\infty \frac{\phi^2}{r^2}\left[\frac{\eta(u)-\eta(v)}{u-v}\right] r^{2n-1}dr \\
&\leq -\|\phi_r\|_{L^2}^2 + \frac{(d-2)^2}{4}\left\|\frac{\phi}{r}\right\|_{L^2}^2 \\
&\leq -\|\phi_r\|_{L^2}^2 + \frac{(d-2)^2}{4}\frac{4}{(d-2)^2}\|\phi_r\|_{L^2}^2 \leq 0, \qquad (2.5.7)
\end{aligned}$$

where in the last line we have used Hardy's inequality with the best constant $4/(d-2)^2$. This implies that $\phi \equiv 0$, and hence that $u = v$.

We now show how the condition $u_0 \in L^2$ may be removed so that uniqueness holds for all data u_0 simply in L^∞. The following is, up to a permutation of notation, an argument of [23]. We repeat it here merely for completeness and in no way to claim it as our own.

Fix $\epsilon > 0$ and define,

$$\psi^\epsilon(t,x) = \phi(t,x) - \epsilon\langle x\rangle,$$

where $\langle x\rangle = \sqrt{1+|x|^2}$ and $\phi = u - v$ is as before. The function ψ^ϵ satisfies the equation,

$$\begin{aligned}
\psi_t^\epsilon - \Delta\psi^\epsilon &= \phi_t - \Delta\phi + \epsilon\Delta\langle x\rangle \\
&= \frac{1}{r^2}(-\eta(u)+\eta(v)) + \epsilon\Delta\langle x\rangle \\
&\leq \frac{1}{r^2}\frac{(d-2)^2}{4}|u-v| + \epsilon\frac{C}{r} \\
&\leq \frac{1}{r^2}\frac{(d-2)^2}{4}|\psi^\epsilon| + \epsilon\frac{C}{r}. \qquad (2.5.8)
\end{aligned}$$

Define $\psi_+^\epsilon = \max(\psi^\epsilon, 0)$. From the definition of ψ^ϵ we see that $\operatorname{supp} z_+^\epsilon \subset B(0, R/\epsilon)$, where $R = 2\|\psi^\epsilon\|_{L^\infty}$.

We now calculate using the inequality (2.5.8),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi_+^\epsilon\|_{L^2}^2 &= \int_{\mathbb{R}^d} \psi_+^\epsilon (\psi_+^\epsilon)_t \, dx \\ &\le \int_{\mathbb{R}^d} \psi_+^\epsilon (\Delta \psi_+^\epsilon) dx + \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{(\psi_+^\epsilon)^2}{r^2} dx + C\epsilon \int_{\mathbb{R}^d} \frac{\psi_+^\epsilon}{r} dx \\ &\le -\|\nabla \psi_+^\epsilon\|_{L^2}^2 + \|\nabla \psi_+^\epsilon\|_{L^2}^2 + C\epsilon \int_{\mathbb{R}^d} \frac{\psi_+^\epsilon}{r} dx,\end{aligned}$$

where in the last step we used Hardy's inequality with optimal constant. The first two terms on the right hand side cancel. For the last term we have, by the support property of ψ_+^ϵ and the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi_+^\epsilon\|_{L^2}^2 &\le C\epsilon \int_{\mathbb{R}^d} \frac{\psi_+^\epsilon}{r} dx = C\epsilon \int_{B(0,R/\epsilon)} \frac{\psi_+^\epsilon}{r} dx \\ &\le C\epsilon \|\psi_+^\epsilon\|_{L^2} \left(\int_{B(0,R/\epsilon)} \frac{1}{r^2} dx \right)^{1/2} \le C\sqrt{\epsilon} + C\sqrt{\epsilon}\|\psi_+^\epsilon\|_{L^2}^2.\end{aligned}$$

This differential inequality implies that,

$$\|\psi_+^\epsilon\|_{L^2}^2 \le e^{C\sqrt{\epsilon}t} - 1,$$

which gives, as $\epsilon \to 0$, that $\max(\phi, 0) = 0$. A similar argument shows that $\min(\phi, 0) = 0$ and hence that $\phi = u - v = 0$. □

In this context of the real equivariant heat flow from $\mathbb{C}^n$ to $\mathbb{CP}^n$, this implies the following result (given as Theorem 5 (i) in the introduction).

Proposition 2.5.2. *Let $n \ge 3$. For a given initial data there is at most one solution to* (2.5.2) *in $L_t^\infty L_x^\infty$.*

Proof. Here $d = 2n$ and $\eta(x) = (2n-2)\sin(x) + \sin(2x)/2$. We calculate,

$$\begin{aligned}\eta'(x) &= (2n-2)\cos(x) + 2\cos^2(x) - 1 \\ &= (2n-6)\cos(x) + 2(\cos(x)+1)^2 - 3 \\ &\ge (2n-6)(-1) + 0 - 3 = -(2n-3),\end{aligned}$$

where the last inequality holds because $n \ge 3$ and so $(2n-6) \ge 0$. Now using the inequality $-(2n-3) \ge -(n-1)^2$ (which is equivalent to $3 \ge -(n+1)^2$) gives condition (2.5.6) and hence the result. □

2.5.2 · THE $\mathbb{CP}^2$ CASE: BREAKDOWN OF UNIQUENESS

The $n = 2$ case is the most interesting. From the expression,

$$\eta'(x) = 2\cos(x) + \cos(2x),$$

we see that $\eta'(\pi) = -1$, which is precisely the threshold $-(d-2)^2/4 = -1$ in the conditions (2.5.5) and (2.5.6). The condition that would imply non-uniqueness, (2.5.5), does not hold. However we find that,

$$\eta''(\pi) = -2\sin(\pi) - 4\sin(2\pi) = 0,$$

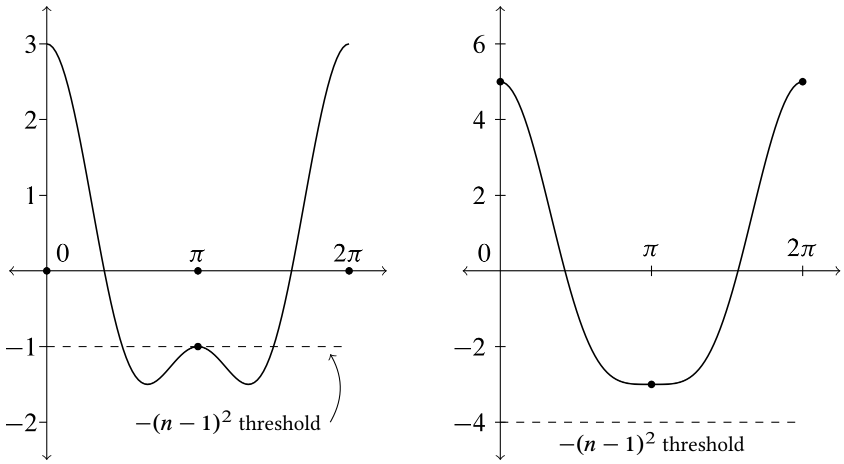

Figure 2.2: Plots of the function $\eta'(x)$ in the case of the real equivariant heat flow from $\mathbb{C}^n$ to $\mathbb{CP}^n$ in the cases $n = 2$ (left) and $n = 3$ (right). For the $n = 3$ case, we easily see that η satisfies the condition (2.5.6) with $d = 2n$, and hence that uniqueness in $L_t^\infty L_x^\infty$ holds. For the $n = 2$ case, we see that both (2.5.5) and (2.5.6) do not hold, so the case does not fit into the general classification theory.

and,

$$\eta'''(\pi) = -2\cos(\pi) - 8\cos(2\pi) = -2(-1) - 8(+1) = -6 < 0,$$

so in fact, by the second derivative test, π is a local *maximum* of $\eta'(x)$. This means that the condition that would imply uniqueness, (2.5.6), does hold either. Hence the case of the real equivariant heat flow from $\mathbb{C}^2$ to $\mathbb{CP}^2$ is a borderline case not covered by the classification theorem of [26]. (Plots of η in the $n = 2$ and $n = 3$ cases are given in Figure 2.2, which make the difference clear.)

The question is then: does uniqueness hold or not? First, we see that the proof of uniqueness presented in the last section clearly breaks down: because the derivative goes below the threshold value $-(d-2)^2/4$, a Lipshitz inequality of the form $\eta(u) - \eta(v)/(u - v) \geq -(d-2)^2/4$ cannot hold.

On the other hand, inspecting the proof in [26] of non-uniqueness in the case (2.5.5) we see that it relies critically on the following fact: if condition (2.5.5) holds, then the stationary solutions (that is, the harmonic maps) of the PDE problem oscillate around the fixed point π as they converge to it. In our case, the harmonic maps are given explicitly by $\psi_\alpha(r) = 2\arctan(\alpha r)$ and are clearly not oscillatory, and so that proof of non-uniqueness will not hold. In fact, what is interesting is that the harmonic maps being monotonic is ordinarily a sign that there is uniqueness (if the uniqueness condition (2.5.6) holds, then the harmonic maps are necessarily monotonic.) However, by using an alternative method in [26] we are able to show that uniqueness for the problem from $\mathbb{C}^2$ to $\mathbb{CP}^2$ does not hold. The original theorem requires some background to state, so we state a special version adapted to our setting.

Theorem ([26], Theorem 2.2)**.** *Suppose that the 'equator map' $u(r,t) \equiv \pi$ (which is a time independent solution of the PDE) does not minimize the energy,*

$$E(f) = \int_0^1 \left[|f'|^2 + \frac{\gamma(f)}{r^2} \right] r^{d-1} dr,$$

where $\gamma'(x) = \eta(x)$. Then there exists a self-similar weak solution of the initial value problem (2.5.4) *that is not constant in time and that has the same initial data as the equator map, $u_0(r) \equiv \pi$.*

Using this, we prove part (ii) of Theorem 5 in the introduction.

Proposition 2.5.3. *For the case $n = 2$ there is non-uniqueness of the problem* (2.5.2)*: there are two distinct solutions with initial data $u_0(r) \equiv \pi$.*

Proof. The key aspect of the proof is capturing the fact that in the $n = 2$ case, the condition $\eta'(x) \geq -(d-2)^2/4 = -1$ in (2.5.6) is violated. If the non-uniqueness condition $\eta'(\pi) < -1$ in (2.5.5) held, this would be easy. However because $\eta'(\pi) = -1$, we need to do a higher order expansion of $\eta'(x)$ around π to show this. Once we establish that condition $\eta'(x) \geq -1$ is violated, we follow [41] and construct h based on a function which almost saturates that Hardy inequality.

Let $u = \pi$ denote the equator map and h be any function. We have,

$$E(h) - E(u) = \int_0^1 \left[|h'|^2 + \frac{\gamma(h) - \gamma(\pi)}{r^2} \right] r^{d-1} dr, \tag{2.5.9}$$

where $\gamma'(x) = \eta(x)$. One calculates,

$$\begin{aligned} \gamma'(\pi) &= \eta(\pi) = 0; \\ \gamma''(\pi) &= \eta'(\pi) = -1; \\ \gamma'''(\pi) &= \eta''(\pi) = 0; \\ \gamma''''(\pi) &= \eta'''(\pi) = -6. \end{aligned}$$

Therefore by a Taylor expansion, if we choose δ small then there exists a constant $C > 0$ such that,

$$\gamma(x) - \gamma(\pi) \leq -(x-\pi)^2 - C(x-\pi)^4, \tag{2.5.10}$$

for all $x \in [\pi - \delta, \pi + \delta]$. The constant C is positive because $\gamma^{(4)}(\pi) < 0$.

To use the inequality (2.5.10) in the energy expression (2.5.9), we need to choose h valued in $[\pi - \delta, \pi + \delta]$.

Following [41], we define, for any $\epsilon > 0$, the function $f_\epsilon : [0, 1] \to \mathbb{R}$ by,

$$f_\epsilon(r) = \begin{cases} \epsilon^{-1} & \text{for } 0 \leq r \leq \epsilon, \\ r^{-1} & \text{for } \epsilon \leq r \leq 1/2, . \\ 4(1-r) & \text{for } 1/2 \leq r \leq 1. \end{cases} \tag{2.5.11}$$

One verifies that $f(r)$ satisfies,

$$\int_0^1 \left| \frac{f}{r} \right|^2 r^3 dr \leq \int_0^1 \left| f' \right|^2 r^3 dr \leq \left(1 + \frac{B}{|\log(\epsilon)|} \right) \int_0^1 \left| \frac{f}{r} \right|^2 r^3 dr, \tag{2.5.12}$$

for some $B > 0$ independent of ϵ. That is, f is close to saturating the Hardy inequality, which in this case has best constant $4/(d-2)^2 = 1$. Then set,

$$h(r) = \pi - \delta \frac{f_\epsilon(r)}{\|f\|_{L^\infty}} = \pi - \delta \frac{f_\epsilon(r)}{2}.$$

We observe that $h(r) \in [\pi - \delta, \pi + \delta]$ for all r.

We then have,

$$
\begin{aligned}
E(h) - E(u) &= \int_0^1 \left[|h'|^2 + \frac{\gamma(h) - \gamma(\pi)}{r^2} \right] r^{d-1} dr \\
&\leq \int_0^1 \left[|h'|^2 + \frac{-(h-\pi)^2 - C(h-\pi)^4}{r^2} \right] r^{d-1} dr \\
&= \int_0^1 \left[\frac{\delta^2}{4} |f_\epsilon'|^2 - \frac{\delta^2}{4} \left| \frac{f_\epsilon}{r} \right|^2 - C \frac{\delta^4}{16} \left| \frac{f_\epsilon}{r} \right|^2 |f_\epsilon|^2 \right] r^{d-1} dr.
\end{aligned}
$$

Now using bound (2.5.12) we determine that,

$$
E(h) - E(u) \leq \frac{\delta^2}{4} \int_0^1 \left| \frac{f_\epsilon}{r} \right|^2 \left(\frac{B}{|\log \epsilon|} - C \frac{\delta^2}{4} |f_\epsilon|^2 \right) r^3 dr,
$$

and by choosing ϵ sufficiently small we may make the right hand side negative.

We thus determine that $E(h) < E(u)$, and hence there are two solutions. □

2.5.3 · The $n \geq 3$ case: precise dynamics of the self similar solutions

We finally present some results on the dynamics of the self-similar solutions in the real heat flow case when $n \geq 3$. The methods of analysis here are not original, and our results are based on analogous results elsewhere. Our motivation in presenting them here to show how in this special case, one can determine precise dynamics of the self-similar solutions; it would be very satisfactory to extend these results to the general case of the GLL equation.

We first recall the self-similar problem.

Definition 2.5.3. The self-similar real heat flow problem is the ODE,

$$
0 = \phi_\beta''(r) + \left(\frac{2n-1}{r} + \frac{r}{2} \right) \phi_\beta'(r) - \frac{1}{r^2} \eta(\phi_\beta),
$$

subject the initial conditions $\phi_\beta(0) = 0$ and $\phi_\beta'(0) = \beta > 0$.

From section 2.3 we know that for every $\beta > 0$ there is a unique global solution to this problem and that there exists $\phi_\beta(\infty) \in \mathbb{R}$ such that $\lim_{r\to\infty} \phi_\beta(r) = \phi_\beta(\infty)$.

Proposition 2.5.4. *Let ϕ_β be the solution of the self-similar problem and ψ_β the solution of the stationary problem.*

(i) We have the bound $\phi_\beta(r) \leq \psi_\beta(r)$.

(ii) The function ϕ_β is monotonically increasing and $\phi_\beta(r) < \pi$.

(iii) For fixed $r > 0$, the function $\beta \mapsto \phi_\beta(r)$ is strictly increasing, $\phi_0(r) = 0$, and $\lim_{\beta\to\infty} \phi_\beta(r) = \pi$.

(iv) The function $\beta \mapsto \phi_\beta(\infty)$ is strictly increasing, $\phi_0(\infty) = 0$, and $\lim_{\beta\to\infty} \phi_\beta(\infty) = \pi$.

The content of this Proposition may be seen at a glance in Figure 2.3. Note that in light of the non-uniqueness theorem for $n = 2$, we don't expect the same dynamics in the $n = 2$ case: in fact we expect a self-similar profile whose asymptotic limit is π.

Lemma 2.5.5. *Suppose that for all $r \in [0, R]$, we have $\phi_\beta(r) < \pi$. Then ϕ_β is increasing on $[0, R]$.*

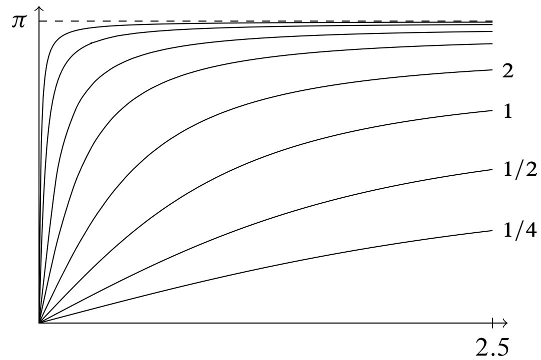

Figure 2.3: Plots of $\phi_\beta(r)$ for $r \in [0, 2.5]$ and $\beta = 0.25, 0.5, 1, 2, 4.5, 10, 30$ and 100.

Proof of lemma. Becase $\beta > 0$, the solution is initially increasing. For a contradiction, let r_0 be the first critical point in $[0, R]$. Because ϕ_β is initially increasing, r_0 must be a local maximum. However from the ODE we have,

$$\phi_\beta''(r_0) = -\left(\frac{2n-1}{r} + \frac{r}{2}\right)\phi_\beta'(r_0) + \eta(\phi_\beta(r_0)) = \eta(\phi_\beta(r_0)) > 0,$$

where $\eta(\phi_\beta(r_0)) > 0$ because $\phi_\beta(r_0) \in (0, \pi)$. The condition $\phi_\beta''(r_0) > 0$ contradicts r_0 being a maximum. Hence ϕ_β is increasing on $[0, R]$. □

Proof of Proposition 2.5.4, (i). Let $\epsilon > 0$ and consider the functions ϕ_β and $\psi_{\beta+\epsilon}(r)$. Define,

$$f(r) = r^2(\psi_{\beta+\epsilon}(r) - \phi_\beta(r)).$$

We will show that $f(r) \geq 0$ for all r. Letting $\epsilon \to 0$ will then give the result.

By continuity of derivatives given by the well-posedness theory, there is an initial interval $[0, \delta)$ on which $\psi_{\beta+\epsilon}(r) - \phi_\beta(r)$ is increasing, and hence, as r^2 is also increasing, the function f is increasing on this interval.

Now suppose that f has a critical point. Let r_0 be the first critical point. Because f is initially increasing, this critical point must be a local maximum. Because f is increaing on $(0, r_0)$, we have $f(r_0) > 0$.

We then calculate,

$$\begin{aligned} f''(r) &= r^2(\psi_{\beta+\epsilon}''(r) - \phi_\beta''(r)) + 4r(\psi_{\beta+\epsilon}'(r) - \phi_\beta'(r)) + 2(\psi_{\beta+\epsilon}(r) - \phi_\beta(r)) \\ &= \frac{4-(2n-1)}{r} f' + \frac{2(2n-1)-6}{r^2} f + \frac{r^3}{2}\phi_\beta' + \eta(\psi_{\beta+\epsilon}) - \eta(\phi_\beta). \end{aligned} \tag{2.5.13}$$

Firstly, we have the Lipshitz bound,

$$\eta(\psi_{\beta+\epsilon}(r_0)) - \eta(\phi_\beta(r_0)) \geq -(2n-3)(\psi_{\beta+\epsilon}(r_0) - \phi_\beta(r_0)),$$

where we have used the fact that $f(r_0) = \psi_{\beta+\epsilon}(r_0) - \phi_\beta(r_0) > 0$ to multiply across by $\psi_{\beta+\epsilon}(r_0) - \phi_\beta(r_0)$.

Secondly, because $f(r_0) > 0$, $\phi_\beta(r_0) < \psi_{\beta+\epsilon}(r_0) < \pi$, and hence by the Lemma ϕ_β is increasing on $[0, r_0]$. Therefore $\phi_\beta'(r_0) \geq 0$.

Using both of these inequalities, and as well as $f'(r_0) = 0$, in (2.5.13) yields,

$$\begin{aligned} f''(r_0) &\geq +\frac{2(2n-1)-6}{r_0^2} f(r_0) + 0 - \frac{2n-3}{r_0^2} f(r_0) \\ &= \frac{2n-5}{r_0^2} f(r_0) > 0, \end{aligned}$$

which contradicts r_0 being a local maximum. Hence f has no critical points; it is increasing for all r. In particular, it is always positive, so $\phi_\beta(r) < \psi_{\beta+\epsilon}(r)$ for all r. Taking the limit $\epsilon \to 0$ then gives $\phi_\beta(r) \leq \psi_\beta(r)$. □

Proof of Proposition 2.5.4, (ii). The previous bound gives $\phi_\beta(r) \leq \psi_\beta(r) < \pi$ for all r. Hence by the Lemma, $\phi_\beta(r)$ is always increasing. □

Proof of Proposition 2.5.4, (iii). Set $\alpha < \beta$. We wish to show that $\phi_\alpha(r) < \phi_\beta(r)$, which follows from a maximum principle analysis of $g(r) = r^2(\phi_\beta(r) - \phi_\alpha(r)$. The analysis is similar to the proof of item 2. The function g is is initially increasing. If r_0 denotes the first critical point, which must be a maximum, one calculates,

$$\begin{aligned} g''(r_0) &= \left[\frac{4n-8}{r^2} + 1\right] g(r_0) + \frac{\phi_\beta(r_0) - \phi_\alpha(r_0)}{r^2} \\ &\geq \left[\frac{4n-8}{r^2} + 1\right] g(r_0) - \frac{2n-3}{r^2} g(r_0) = \left[\frac{2n-5}{r^2} + 1\right] g(r_0) \geq 0, \end{aligned}$$

a contradiction. Therefore g is increasing for all r, and in particular is positive, and hence $\psi_\beta(r) > \psi_\alpha(r)$. □

Proof of Proposition 2.5.4, (iv). The proof follows from a similar maximum principle argument as in the previous proof to show that the function,

$$h(r) = \left(\frac{r}{2+r}\right)^2 (\psi_\beta(r) - \psi_\alpha(r)),$$

is increasing. One then has, for $r > 1$,

$$\left(\frac{r}{2+r}\right)^2 (\psi_\beta(r) - \psi_\alpha(r)) \geq \frac{1}{9}(\psi_\beta(1) - \psi_\alpha(1)) > 0,$$

and hence on taking limits,

$$(\psi_\beta(\infty) - \psi_\alpha(\infty)) \geq \frac{1}{9}(\psi_\beta(1) - \psi_\alpha(1)) > 0,$$

which is what we wanted to prove. □

2.5.4 · HARDY INEQUALITIES

Theorem 2.5.6 (Generalized radial Hardy inequality)**.** *Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is radial. Then for all $p \geq 1$ and $k \geq 0$ such that $p < d/(k+1)$ there holds,*

$$\left\| \frac{f}{r^{k+1}} \right\|_{L^p} \leq \frac{p}{d - p(k+1)} \left\| \frac{f_r}{r^k} \right\|_{L^p}. \tag{2.5.14}$$

Proof. We suppose that f is smooth and compactly supported. The result for arbitrary f then follows from a standard density argument.

We have,

$$\frac{d}{dr}\left(\frac{f}{r^k}\right) = -k\frac{f}{r^{k+1}} + \frac{f_r}{r^k}.$$

Multiplying this equation by $(f/r^{k+1})^{p-1} r^{d-1}$ and integrating over $[0, \infty$ yields,

$$\int_0^\infty \frac{d}{dr}\left(\frac{f(r)}{r^k}\right)\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1} = -\frac{k}{s(d)}\left\|\frac{f}{r^{k+1}}\right\|_{L^p}^p + \int_0^\infty \frac{f_r}{r^k}\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1},$$

where $s(d)$ is the measure of the unit sphere in $\mathbb{R}^d$. Now performing integration by parts on the term on the left we find,

$$\begin{aligned}\int_0^\infty \frac{d}{dr}\left(\frac{f}{r^k}\right)&\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1} \\ &= -\int_0^\infty \left(\frac{f}{r^k}\right)\frac{d}{dr}\left[\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1}\right] dr + \left[\left(\frac{f}{r^{k+1}}\right)^p r^d\right]\Bigg|_{r=0}^{r=\infty}.\end{aligned}$$

The boundary term corresponding to $r = \infty$ is 0 because f is compactly supported. For the $r = 0$ term we find,

$$\lim_{r\to 0}\left(\frac{f}{r^{k+1}}\right)^p r^d = \lim_{r\to 0} f(r)^p r^{d-p(k+1)} = 0,$$

if $d - p(k+1) > 0$. We therefore have,

$$\begin{aligned}
&\int_0^\infty \frac{d}{dr}\left(\frac{f}{r^k}\right)\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1} \\
&\qquad = -\int_0^\infty \left(\frac{f}{r^k}\right)\frac{d}{dr}\left[f(r)^{p-1}r^{d-1-(p-1)(k+1)}\right]dr \\
&\qquad = -\int_0^\infty \left(\frac{f}{r^k}\right)\Big[(d-1-(p-1)(k+1))f(r)^{p-1}r^{d-2-(p-1)(k+1)} + \\
&\qquad\qquad\qquad +(p-1)f(r)^{p-2}f_r(r)r^{d-1-(p-1)(k+1)}\Big]dr \\
&\qquad = -\frac{(d-p(k+1)+k)}{s(d)}\left\|\frac{f}{r^{k+1}}\right\|_{L^p}^p - (p-1)\int_0^\infty \frac{f_r}{r^k}\left(\frac{f}{r^{k+1}}\right)^{p-1} r^{d-1}.
\end{aligned}$$

Substituting this into the equation above and combining terms we get,

$$(d-p(k+1))\left\|\frac{f}{r^{k+1}}\right\|_{L^p}^p = -s(d)p\int_{\mathbb{R}^d}\frac{f_r}{r^k}\left(\frac{f}{r^{k+1}}\right)^{p-1}dx \le p\left\|\frac{f_r}{r^{k+1}}\right\|_{L^p}\left\|\frac{f}{r^{k+1}}\right\|_{L^p}^{p-1},$$

which upon dividing through by the norm of f/r^{k+1} gives the result. □

Corollary 2.5.7. *Suppose that $f : \mathbb{R}^d \to X$ is radial with $X = \mathbb{C}$ or $X = \mathbb{R}^m$. Then for all $p \ge 1$ and $k \ge 0$ such that $p < d/(k+1)$ there is a constant $C(d,p,X)$ such that,*

$$\left\|\frac{f}{r^{k+1}}\right\|_{L^p} \le C(d,p,X)\left\|\frac{f_r}{r^k}\right\|_{L^p}. \tag{2.5.15}$$

Proof. Take $X = \mathbb{C}$ and write f as $f(r) = a(r) + ib(r)$ for real valued functions a and b. Using that $\|u\|_{L^p} \sim \|\mathrm{Re}\, u\|_{L^p} + \|\mathrm{Im}\, u\|_{L^p}$, we have,

$$\left\|\frac{f}{r^{k+1}}\right\|_{L^p} \lesssim \left\|\frac{a}{r^{k+1}}\right\|_{L^p} + \left\|\frac{b}{r^{k+1}}\right\|_{L^p} \lesssim \left\|\frac{a_r}{r^k}\right\|_{L^p} + \left\|\frac{b_r}{r^k}\right\|_{L^p} \lesssim \left\|\frac{f_r}{r^k}\right\|_{L^p}.$$

A similar argument holds $X = \mathbb{R}^m$ writing f in terms of its real-valued coordinate functions. □

2.5.5 · Local wellposedness for a class of singular ODE

Theorem 2.5.8. *Consider the Cauchy problem,*

$$\begin{aligned}
f''(r) &= A\left(f'(r), f(r), r\right) - k\left(\frac{f'(r)}{r} - \frac{f(r)}{r^2}\right) + \frac{1}{r^2}B(f(r)), \qquad (2.5.16)\\
f(0) &= 0,\\
f'(0) &= \alpha \in \mathbb{C},
\end{aligned}$$

where,

- $k > 0$,
- $A(z_1, z_2, r)$ *is a smooth function with* $A(\alpha, 0, 0) = 0$,

- *$B(z)$ is a smooth function such $|B(z)| \leq C|z|^3$ in a neighbourhood of 0, and $(\partial B/\partial z)(0) = (\partial B/\partial \overline{z})(0) = 0$.*

There exists $r_0 > 0$ such that there is a unique solution among all functions $f : [0, r_0] \to \mathbb{C}$ satisfying,

$$|f(r)|_{L^\infty([0,r_0])} + \left| \frac{f'(r) - f'(0)}{r} \right|_{L^\infty([0,r_0])} < \infty. \tag{2.5.17}$$

The unique solution in this space is second differentiable at $r = 0$ and satisfies $f''(0) = 0$.

Let us make two remarks on the conditions in the theorem.

- The condition (2.5.17) on f is equivalent to both f and f' belonging to L^∞ and f' satisfying a Liphitz condition at $r = 0$.
- The assumptions on B ensure that its behaviour as $r \to 0$ is non-singular; indeed, one readily verifies that, for smooth f, $B(f(r))/r^2 \to 0$ as $r \to 0$. With this formulation of the Cauchy problem the singular behavior occurs only in the term $\kappa(f'(r)/r - f(r)/r^2)$.

Proof. Step one: setting up the fixed point formulation. For any $r > 0$, define μ_r by,

$$\mu_r(f) = \|f(r)\|_{L^\infty([0,r])} + \left\| \frac{f'(r) - f'(0)}{r} \right\|_{L^\infty([0,r])},$$

and let $X_{r,M}$ denote all functions f such that $\mu_r(f) \leq M$. On the space $X_{r,M}$ we define a distance function for $f, g \in X_{r,M}$ by,

$$d_r(f,g) = \|f - g\|_{L^\infty([0,r])} + \left\| \frac{f' - g'}{r} \right\|_{L^\infty([0,r])}.$$

We will perform the fixed point argument in the complete metric space $(X_{r,M}, d_r)$.

We now express the Cauchy problem as a fixed point problem for an operator defined on $X_{r,M}$. Multiply equation (2.5.16) by r^k and integrate from 0 to r. Using integration by parts, the left hand side is,

$$\int_0^r f''(s)s^k ds = f'(r)r^k - k\int_0^r f'(s)s^{k-1}ds,$$

while the right hand side is,

$$\int_0^r \left[A(f'(s), f(s), s) + \frac{B(f(s))}{s^2} - k\frac{f'(s)}{s} + k\frac{f(s)}{s^2} \right] s^k ds.$$

We see that the integral terms involving $kf'(s)$ cancel, giving,

$$f'(r) = \frac{1}{r^k}\int_0^r \left[A(f'(s), f(s), s)s^k + B(f(s))s^{k-2} + k\frac{f(s)}{s}s^{k-1} \right] ds.$$

Using the fact that $\frac{1}{r^k}\int_0^r ks^{k-1}ds = 1$, we can write this as,

$$f'(r) = \frac{1}{r^k}\int_0^r \Big[A(f'(s), f(s), s)s^k + B(f(s))s^{k-2} + k\left(\frac{f(s)}{s} - f'(0)\right)s^{k-1} \Big] ds + f'(0)$$

To find an equation for $f(r)$ we simply integrate this equation from 0 to r.

We therefore define T by,

$$(Tf)(r) = \int_0^r \left\{ \frac{1}{t^k} \int_0^t \left[A(f'(s), f(s), s)s^k + B(f(s))s^{k-2} \right.\right. \\ \left.\left. + k\left(\frac{f(s)}{s} - f'(0)\right)s^{k-1} \right] ds \right\} dt + rf'(0).$$

By the preceding computations, constructing solutions of the Cauchy problem (2.5.16) is equivalent to finding a fixed point of T.

Step two: T maps a space into itself. We will show that for any M, there exists r_0 such that T maps $X_{r,M}$ into $X_{r,M}$ for all $r \leq r_0$.

Fix $M > 0$. We will assume throughout that $r \leq r_0 \leq 1$, as we are only interested in the local theory about $r = 0$. Thus if $\mu_{r_0}(f) \leq M$ we have for all $r \leq r_0$,

$$|f(r)| \leq \mu_r(f) \leq M \quad \text{and} \quad |f'(r)| \leq |f'(0)| + r\mu_r(f) \leq |f'(0)| + M; \tag{2.5.18}$$

that is to say, if $\mu_r(f) \leq \mu_{r_0}(f) \leq M$ then $f(r)$ and $f'(r)$ are valued in a bounded set in the complex plane for all $r \leq r_0$. By the smoothness conditions on A and B we therefore have that that there exists a constant C_M depending only on M such that for every $f \in X_{r,M}$,

$$|A(f'(r), f(r), r)| \leq C_M \left(|f'(r) - f'(0)| + |f(r)| + r\right) \tag{2.5.19}$$

$$|B(f(r))| \leq C_M |f(r)|^3. \tag{2.5.20}$$

In what follows the constant C_M may change from line-to-line, but in all cases indicates a constant that only depends on M and $\alpha = f'(0)$.

We first examine $(Tf)'(r)$. It is clear from the definition of T that if $f \in X_{r,M}$ then $(Tf)'(0) = f'(0)$. We therefore have,

$$\left| \frac{(Tf)'(r) - (Tf)'(0)}{r} \right| \leq \frac{1}{r^{k+1}} \int_0^r \left[|A(f'(s), f(s), s)|s^k \right. \\ \left. + |B(f(s))|s^{k-2} + k\left|\frac{f(s)}{s} - f'(0)\right| s^{k-1} \right] ds$$

We deal with each of the three terms on the right hand side in turn.

Using (2.5.19), the first term is,

$$\begin{aligned} \frac{1}{r^{k+1}} \int_0^r |A(f'(s), f(s), s)|s^k ds &\leq C_M \frac{1}{r^{k+1}} \int_0^r \left(|f'(s) - f'(0)| + |f(s)| + s\right) s^k ds \\ &= C_M \frac{1}{r^{k+1}} \int_0^r \left(\frac{|f'(s) - f'(0)|}{s} s + |f(s)| + s\right) s^k ds \\ &\leq C_M \frac{1}{r^{k+1}} \int_0^r \left(\mu_r(f)s + s|f'(0)| + \frac{s^2}{2}\mu_r(f) + s\right) s^k ds, \end{aligned}$$

where in the last step we have used (2.5.18). On evaluating the integrals, and using $r \leq 1$, we find,

$$\frac{1}{r^{k+1}} \int_0^r |A(f'(s), f(s), s)|s^k ds \leq C_M \mu_r(f) r + |f'(0)|r + r.$$

Now for the second term. Using (2.5.20) we have,

$$
\begin{aligned}
\frac{1}{r^{k+1}}\int_0^r |B(f(s))|s^{k-2}ds &\leq C_M \frac{1}{r^{k+1}}\int_0^r |f(s)|^3 s^{k-2}ds \\
&\leq C_M \frac{1}{r^{k+1}}\int_0^r \left(s^3|f'(0)| + s^6\mu_{r_0}(f)\right)s^{k-2}ds \\
&\leq C_M \mu_r(f)r + |f'(0)|r.
\end{aligned}
$$

For the last term, we first look at,

$$
\left|\frac{f(s)}{s} - f'(0)\right| \leq \frac{1}{s}\int_0^s |f'(t) - f'(0)|dt = \frac{1}{s}\int_0^s \frac{|f'(t) - f'(0)|}{t}t\,dt \leq \frac{s}{2}\mu_r(f).
$$

This gives,

$$
\begin{aligned}
\frac{1}{r^{k+1}}\int_0^r k\left|\frac{f(s)}{s} - f'(0)\right|s^{k-1}ds &\leq \frac{k}{2r^{k+1}}\int_0^r \mu_r(f)s^k ds \\
&= \frac{k}{2(k+1)}\mu_r(f) \leq \frac{1}{2}\mu_r(f).
\end{aligned}
$$

With these three estimates we have established that,

$$
\left|\frac{(Tf)'(r) - (Tf)'(0)}{r}\right| \leq C_M r(\mu_r(f) + |f'(0)| + 1) + \frac{1}{2}\mu_r(f) \leq C_M r + \frac{M}{2}. \tag{2.5.21}
$$

The L^∞ estimate on Tf follows immediately from this,

$$
|(Tf)(r)| \leq \int_0^r \frac{|(Tf)'(s) - (Tf)'(0)|}{s}s\,ds + rf'(0) \leq C_M r. \tag{2.5.22}
$$

With the two estimates (2.5.21) and (2.5.22) we thus have,

$$
\mu_r(Tf) \leq C_M r + \frac{M}{2},
$$

for all $r \leq r_0$. By choosing r_0 sufficiently small (namely $r_0 \leq M/2C_M$), we have that for all $r \leq r_0$ that $\mu_{r_0}(Tf) \leq M$, and hence that $T : X_{r,M} \to X_{r,M}$.

Step three: T is a contraction mapping. We will show that for sufficiently small r_0, T is a contraction mapping in $X_{r_0,M}$.

As before, fix M. For sufficiently small r_0, T maps $X_{r,M}$ into itself for every $r \leq r_0$. Take $f, g \in X_{r,M}$. By the smoothness assumptions there is a constant C_M such that

$$
|A(f'(r), f(r), r) - A(g'(r), (r), r)| \leq C_M \left(|f(r) - g(r)| + |f'(r) - g'(r)|\right) \tag{2.5.23}
$$

$$
|B(f(r)) - B(g(r))| \leq C_M |f(r) - g(r)| \left(|f(r)| + |g(r)|\right), \tag{2.5.24}
$$

for all $r \leq r_0$.

Again we look at the derivative term first. We have,

$$\left|\frac{(Tf)'(r)-(Tg)'(r)}{r}\right| \leq \frac{1}{r^{k+1}}\int_0^r \Big[|A(f'(s),f(s),s)-A(g'(s),g(s),s)|s^k + |B(f(s))-B(g(s))|s^{k-2} + k\left|\frac{f(s)-g(s)}{s}\right|s^{k-1}\Big]\,ds.$$

By similar arguments to before, and using conditions (2.5.23) and (2.5.24), we determine that,

$$\left|\frac{(Tf)_2(r)-(Tg)_2(r)}{r}\right| \leq C_M r d_r(f,g) + \frac{1}{2}d_r(f,g).$$

Again the L^∞ estimate is a direct consequence of this,

$$|(Tf)(r)-(Tg)(r)| = \left|\int_0^r \frac{(Tf)'(s)-(Tg)'(s)}{s} s\,ds\right| \leq C_M r d_r(f,g).$$

The previous two equations imply that,

$$d_r(Tf,Tg) \leq C_M r d_r(f,g) + \frac{1}{2}d_r(f,g).$$

Hence choosing r_0 sufficiently small, T is a contraction on $X_{r_0,M}$. It follows from Banach's fixed point theorem there exists is a unique solution of the Cauchy problem (2.5.16) in $X_{r_0,M}$.

Note that the constant M in the proof was arbitrary. Setting $M=1$, say, gives us one solution f of the Cauchy problem that is unique in the set $X_{r_0,1}$ for r_0 sufficiently small. Now suppose $\tilde{f}$ is another solution of the Cauchy problem satisfying (2.5.17). This means $f \in X_{r_0',M}$ for some r_0' and some M. We may assume that $M>1$, that $r_0' \leq r_0$, and that r_0' is such that the existence and uniqueness argument above applies. We then have $X_{r_0',1} \subset X_{r_0',M}$, and so f and $\tilde{f}$ are both in the space $X_{r_0',M}$. Because the solution in this space is unique, $f=\tilde{f}$.

Step four: the second derivative at $r=0$ is 0.

The previous steps gives that for $M=1/n$ there exists r_n such that the unique solution f of the Cauchy problem (2.5.16) is in $X_{r_n,1/n}$. We may assume that $r_n \leq 1/n$. In an equation, this says,

$$|f(r)|_{L^\infty([0,r_n])} + \left|\frac{f'(r)-f'(0)}{r}\right|_{L^\infty([0,r_n])} < \frac{1}{n},$$

for some $r_n \leq 1/n$. Taking the limit as $n \to 0$ we see that $f''(0)$ must exist and equal 0. □

2.5.6 · An integration inequality

Proposition 2.5.9. *Suppose that $A'(r)+c_1 rA(r) \leq c_2 r^{-k}$ for $c_1>0$. Then for any $r_0>0$, $A(r) \leq C(c_1,r_0)(A(r_0)e^{-c_1 r^2/4} + c_2 r^{-k+1})$.*

Proof. We may write the equation as,

$$\frac{d}{dr}\left(e^{c_1 r^2/2}A(r)\right) \leq \frac{c_2}{r^k}e^{c_1 r^2/2},$$

which on integration gives,

$$
\begin{aligned}
A(r) &\le e^{c_1(r_0^2 - r^2)/2} A(1) + c_2 e^{-c_1 r^2/2} \int_{r_0}^{r} \frac{1}{s^k} e^{c_1 s^2/2} ds \\
&= e^{c_1(r_0^2 - r^2)/2} A(1) + c_2 \frac{1}{r^{k-1}} \left(\frac{1}{r^{-k+1} e^{c_1 r^2/2}} \int_{r_0}^{r} \frac{1}{s^k} e^{c_1 s^2/2} ds \right).
\end{aligned}
$$

To prove the result we show that the term in the brackets is bounded independently of r. This term is clearly a continuous function of r. Moreover, we have from the condition $c_1 > 0$,

$$
\lim_{r \to \infty} r^{-k+1} e^{c_1 r^2/2} = \infty \quad \text{and} \quad \lim_{r \to \infty} \int_{r_0}^{r} \frac{1}{s^k} e^{c_1 s^2/2},
$$

which means, by L'Hopital's rule, that,

$$
\begin{aligned}
&\lim_{r \to \infty} \left(\frac{1}{r^{-k+1} e^{c_1 r^2/2}} \int_{r_0}^{r} \frac{1}{s^k} e^{c_1 s^2/2} ds \right) \\
&\quad = \lim_{r \to \infty} \left(\frac{1}{(-k+1) r^{-k} e^{c_1 r^2/2} + c_1 r^{-k+2} e^{c_1 r^2/2}} \cdot \frac{1}{r^k} e^{c_1 r^2/2} \right) = \lim_{r \to \infty} \frac{1}{-k + 1 + c_1 r^2} = 0.
\end{aligned}
$$

We thus have for all $r \in [r_0, \infty)$,

$$
\left(\frac{1}{r^{-k+1} e^{c_1 r^2/2}} \int_{r_0}^{r} \frac{1}{s^k} e^{c_1 s^2/2} ds \right) \le C(r_0, c_1),
$$

which completes the proof. □

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